THE THEOREM OF THOMSON AND TAIT AND NATURAL FAMILIES OF TRAJECTORIES*

BY

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In most dynamical investigations relating to conservative forces, in particular those connected with the principle of least action and the developments of Hamilton and Jacobi, it is essential to group the possible motions according to the value of the constant h representing the total energy. The trajectories corresponding to any given value of h are said to form a *natural*† family. In the case of a particle moving in a three-dimensional conservative field, a natural family consists of ∞^4 curves. Examples are the ∞^4 straight lines of space, corresponding to zero force; and the ∞^4 vertical parabolas whose directrices are located in a fixed horizontal plane, corresponding to gravity assumed constant. In a paper published in the preceding volume of these Transactions,‡ the general geometric character of natural families was expressed in terms of osculating circles; and a certain reciprocity, analogous to that of Scheffers for plane isogonal trajectories, was established.

In the present paper, which may be read independently, we start from the remarkable theorem due to THOMSON and TAIT which states that the ∞^2 curves of a natural family which meet any surface orthogonally are necessarily orthogonal to ∞^1 surfaces, that is, form a normal congruence. Our main object is to show that *this property belongs exclusively to natural families.* The new result may be regarded as a converse of Thomson and Tait's theorem and stated as follows:

If a quadruply-infinite system of curves in space is such that ∞^2 curves of the system meet an arbitrary surface orthogonally and always form a normal congruence (that is, admit ∞^1 orthogonal surfaces), then the system is of the natural type.

Natural families present themselves not only in the study of dynamical tra-

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† The term *natural* was first used in this sense by PAINLEVÉ, Journal de Mathématiques, vol. 10, 1894. Of course no connection is to be inferred with natural or intrinsic coördinates, that is, to invariance under the displacement group.

‡ *Natural families of trajectories: conservative fields of forces*, these Transactions, vol. 10 (1909), pp. 201–219. This will be cited as *earlier paper*.

jectories, but also, for example, in the discussion of brachistochrones, catenaries, and optical path curves.* The many analogies that have been observed between these theories are based on the fact that they all lead to variation problems of the particular form†

$$\int F(x, y, z)\,ds = \text{minimum} \qquad (ds = \sqrt{dx^2 + dy^2 + dz^2}),$$

where F is any point function and ds is the element of length. From the purely mathematical point of view a natural family may be defined as the totality of extremals connected with such a variation problem.

In the kinetic interpretation F represents the speed v, as found from the energy equation

$$v^2 = 2(W + h),$$

where W denotes the work function (negative potential) and the mass is assumed to be unity. The integral above then represents the action, $\int v ds$, and the result is the principle of least action in the form given by Jacobi. The complete theorem of Thomson and Tait‡ is as follows: "If from all points of an arbitrary surface particles not mutually influencing one another be projected normally with the proper velocities [so as to make the sum of the kinetic and potential energies have a given value]; points which they reach with equal actions lie on a surface cutting the paths at right angles." The ∞^1 orthogonal surfaces thus appear as surfaces of equal action.§

In the optical interpretation, which is even more concrete, the function F represents the index of refraction ν, varying from point to point, and the integral $\int \nu ds$ is proportional to the time. We thus have Fermat's principle of least time. Thus the paths of light in an isotropic (not necessarily homogeneous) medium (for example, the atmosphere) form a natural family. Every natural family may be obtained in this way. The ∞^1 surfaces orthogonal to a congruence of paths present themselves as wave fronts (surfaces of equal time).‖

* Cf. APPELL, *Mécanique rationnelle*, vol. 2, p. 216. For other interpretations see the introduction to the author's earlier paper.

† This form is characterized by the fact that the transversal relation reduces to orthogonality. The theorem of Thomson and Tait thus presents itself as a special case of Kneser's general theory of transversals. Cf. BOLZA, *Variationsrechnung*, pp. 131, 691.

‡ *Treatise on natural philosophy* (edition of 1879), part 1, section 332. Cf. DARBOUX, *Théorie des surfaces*, vol. 2, p. 468; APPELL, *Mécanique rationnelle*, vol. 2, p. 431.

§ A second converse which the author has obtained is as follows: If particles are projected normally from a base surface into a conservative field, and if the ∞^2 trajectories generated form a normal congruence, then the initial velocities must be such that the total energy is constant. This is true only "in general." In the case of central forces it is in fact possible to obtain normal congruences for which h is not constant.

‖ Usually the sets of wave surfaces so obtained will not form Lamé families, that is, cannot be regarded as members of triply orthogonal systems. The author has shown that the only natural families leading to Lamé families are those composed of circles; these circles must be orthogonal to a fixed sphere (of real, imaginary, or zero radius). The corresponding optical and dynamical statements are quite simple.

In § 1 the general condition that ∞^2 curves shall form a normal congruence is written out for the case of parametric representation. In our applications the curves are orthogonal to some base surface Σ by assumption. It is therefore expedient to expand the general condition in powers of the parameter t which, starting with the value zero for any point of Σ, varies along the orthogonal curve through that point. The result is then of the form

$$\Omega_1 t + \Omega_2 t^2 + \Omega_3 t^3 + \cdots = 0.$$

For a normal congruence *all* the coefficients vanish. If the leading coefficient Ω_1 vanishes the congruence may be described as "approximately of normal character." If the first n coefficients vanish we describe the congruence as "approximately normal to the nth degree of approximation." A congruence orthogonal to a surface Σ is approximately normal when and only when it is possible to construct ∞^2 curves osculating the given curves at the points of Σ and forming a normal congruence (in the exact sense).

In § 2 the converse theorem stated in italics above is proved. The result is then strengthened in two directions. In the first place it is sufficient to require that the ∞^2 curves, belonging to the quadruply-infinite system, which are orthogonal to an arbitrary surface Σ shall form an approximately normal congruence. The discussion then shows that they necessarily form an exact normal congruence. In the second place it is sufficient to make this demand for suitable manifolds (of course infinite) of base surfaces, instead of for all surfaces.

Special cases of the theorem of Thomson and Tait arise where the selected base Σ degenerates into a point, considered as a small sphere, or into a curve, considered as a thin tube. We show in § 4 that the case of base points is not sufficient to characterize the natural type. Systems of ∞^4 curves exist, which, though not natural, have the property that the curves passing through an arbitrary point always form a normal congruence.

If for every curve as base the corresponding congruence is to be normal, the system is necessarily natural. If only the approximately normal character described above is demanded, the more general class of *velocity systems* is obtained (§ 3). This type was defined in our earlier paper in terms of motion (loci of equal speed) in connection with an arbitrary field of force.* Only for conservative fields are the velocity systems of the natural type. The general velocity system is represented by equations of the form

$$y'' = (\psi - y'\phi)(1 + y'^2 + z'^2), \qquad z'' = (\chi - z'\phi)(1 + y'^2 + z'^2),$$

where ϕ, ψ, and χ are arbitrary point functions, and is characterized by the property that the osculating circles constructed at any point p for the ∞^2 curves

*These Transactions, vol. 10 (1909), p. 376.

passing through that point meet at another point P, thus forming a bundle (property A).*

Natural families are distinguished from other velocity systems by an additional peculiarity, property B, or its equivalent D, of our earlier paper.† In § 4 of the present paper it is shown that this is simply expressed by the requirement that the curves through any point form a normal congruence. *Thus the natural type is characterized by these two properties: the curves through any point admit* ∞^1 *orthogonal surfaces, and the osculating circles constructed at the common point form a bundle.*

In the last section (§ 5) a contribution is made to the general theory of quadruply-infinite systems of curves in space, the only assumption being that ∞^2 curves pass through every point, one in each direction. It is shown that with any system there are associated certain surfaces which when employed as bases give rise to congruences of approximately normal character.‡ These surfaces satisfy a certain partial differential equation of the second order. In the particular case of the velocity type this equation reduces to one of the first order, and in the still more special case of the natural type it vanishes identically.

Although the discussion of the present paper is restricted to ordinary space of three dimensions, the results are, in all probability, valid for higher spaces also. In the case of two dimensions, however, the geometric part of Thomson and Tait's theorem becomes trivial since any system of ∞^1 curves necessarily admits orthogonal curves; the converse proposition is then not true.§

§ 1. Conditions for an exact or approximate normal congruence.

The condition that a congruence of curves, represented in the usual form by

$$\frac{dx}{A(x, y, z)} = \frac{dy}{B(x, y, z)} = \frac{dz}{C(x, y, z)},$$

shall be normal is that the corresponding total equation

$$Adx + Bdy + Cdz = 0$$

shall be exact; this gives the familiar result

$$A(B_z - C_y) + B(C_x - A_z) + C(A_y - B_x) = 0.$$

* The analogous systems in the plane, $y'' = (\psi - y'\varphi)(1 + y'^2)$, were first studied by Scheffers, Leipziger Berichte (1898), in connection with isogonal trajectories.

† Property B states the mutual orthogonality of the three hyperosculating circles existing in any bundle. Property D may be expressed in simplified form as follows: The transformation T from the point p to the point P is such that the three lineal elements at p each of which is converted by T into a cocircular element at P are mutually orthogonal.

‡ In general no exactly normal congruences can be obtained.

§ The results of our earlier paper hold for all dimensions. That the present results remain valid in three-dimensional spaces of constant curvature may be shown by conformal representation.

A congruence of curves defines a field of ∞^3 direction elements in space. If these elements are given in terms of three arbitrary parameters u, v, w (instead of x, y, z) the field takes the form

$$x = X(u, v, w), \quad y = Y(u, v, w), \quad z = Z(u, v, w),$$
$$dx : dy : dz = \bar{X}(u, v, w) : \bar{Y}(u, v, w) : \bar{Z}(u, v, w).$$

The condition for a normal congruence then becomes

$$\bar{X}\{(\bar{Y}XY)-(\bar{Z}ZX)\}+\bar{Y}\{(\bar{Z}YZ)-(\bar{X}XY)\}+\bar{Z}\{(\bar{X}ZX)-(\bar{Y}YZ)\}=0,$$

where the parentheses denote jacobians of the enclosed functions with respect to the parameters, so that, for example,

$$(\bar{Y}XY) \equiv \begin{vmatrix} \bar{Y}_u & \bar{Y}_v & \bar{Y}_w \\ X_u & X_v & X_w \\ Y_u & Y_v & Y_w \end{vmatrix}.$$

In the applications below our congruence admits one orthogonal surface Σ by assumption, and we may take our parameters relative to it as follows: Let u, v denote Gaussian parameters on Σ, and let w be the parameter which varies as the point describes one of the orthogonal curves, assuming it to be zero as the point starts out from the surface. (In particular w might represent the distance from Σ along the curve.) The congruence then takes the form

$$\begin{aligned} X &= x + A_1 w + \tfrac{1}{2} A_2 w^2 + \cdots, \\ Y &= y + B_1 w + \tfrac{1}{2} B_2 w^2 + \cdots, \\ Z &= z + C_1 w + \tfrac{1}{2} C_2 w^2 + \cdots, \\ \bar{X} &= A_1 + A_2 w + \tfrac{1}{2} A_3 w^2 + \cdots, \\ \bar{Y} &= B_1 + B_2 w + \tfrac{1}{2} B_3 w^2 + \cdots, \\ \bar{Z} &= C_1 + C_2 w + \tfrac{1}{2} C_3 w^2 + \cdots, \end{aligned}$$

the last three functions being found by differentiating the first three with respect to w. Here the coefficients are all functions of u and v; for x, y, z are to be replaced by the functions defining the surface Σ; A_1, B_1, C_1 are proportional to the direction cosines of the normal to the surface; and the remaining coefficients are arbitrary.

The condition for a normal congruence, expanded in powers of w, now takes the form

$$\Omega_1 w + \Omega_2 w^2 + \Omega_3 w^3 + \cdots = 0,$$

where Ω_i depends for its value upon the quantities $A_1, B_1, C_1, \cdots, A_{i+1}, B_{i+1}, C_{i+1}$, involving only partial derivatives of the first order. The absolute term neces-

sarily vanishes, since the curves are orthogonal to Σ for which $w = 0$ by assumption.

In case the congruence is normal *all* the coefficients Ω will vanish. If the first coefficient Ω_1 vanishes then the congruence may be said to be normal to the first degree of approximation. In this case the curves may be regarded as orthogonal to some surface infinitesimally close to Σ. In general the congruence will be said to be normal to the nth degree of approximation (or to terms of the nth order) provided

$$\Omega_1 = \Omega_2 = \cdots = \Omega_n = 0 .$$

In the following applications we take up first the general case where Σ is a proper surface, and then the special cases where Σ reduces to a curve and to a point respectively. In the case of surface and curve we shall make use of the explicit value of Ω_1 only; in the case of a point it will appear that Ω_1 vanishes identically and it will be necessary to calculate Ω_2 and Ω_3.

§ 2. General case where the base is a proper surface.

Consider an arbitrary quadruply infinite system of curves in space, assuming that one passes through each point in each direction. Such a system may be defined by a pair of differential equations of the second order

$$y'' = F(x, y, z, y', z'), \qquad z'' = G(x, y, z, y', z'), \tag{1}$$

where F and G are uniform functions which we assume to be analytic in the five arguments. Denoting the initial values of x, y, z, y', z', which may be taken at random, by x, y, z, p, q respectively, and employing X, Y, Z as current coördinates, we may write the solutions of (1) in the form

$$\begin{aligned} Y &= y + p(X-x) + \tfrac{1}{2}F(X-x)^2 + \tfrac{1}{6}M(X-x)^3 + \cdots, \\ Z &= z + q(X-x) + \tfrac{1}{2}G(X-x)^2 + \tfrac{1}{6}N(X-x)^3 + \cdots. \end{aligned} \tag{2}$$

Here F and G are expressed as functions of x, y, z, p, q, and M and N, found by differentiating (1), are given by

$$\begin{aligned} M &\equiv F_x + pF_y + qF_z + FF_p + GF_q, \\ N &\equiv G_x + pG_y + qG_z + FG_p + GG_q. \end{aligned} \tag{3}$$

The terms of higher order will not be needed in our discussion. Equations (2) involve five arbitrary parameters but of course represent only ∞^4 curves.

Consider now an arbitrary surface Σ

$$z = f(x, y). \tag{4}$$

At each point of this surface and normal to it a definite curve of the given family (1) may be constructed. A certain congruence will thus be determined. We wish to express the condition that this shall be of the normal type, that is, that the ∞^2 curves shall admit a family of orthogonal surfaces.

The direction normal to the surface Σ at any point is given by

$$1 : p : q = f_x : f_y : -1,$$

so that

$$p = P(x, y), \qquad q = Q(x, y), \tag{5}$$

where

$$P \equiv \frac{f_y}{f_x}, \qquad Q \equiv -\frac{1}{f_x}. \tag{5'}$$

These functions are connected by the relation

$$PQ_x - QP_x - Q_y \equiv 0. \tag{5''}$$

The equations of the ∞^2 curves corresponding to the given initial conditions may now be written

$$\begin{aligned} Y &= y + Pt + \tfrac{1}{2}\bar{F}t^2 + \tfrac{1}{6}\bar{M}t^3 + \cdots, \\ Z &= f + Qt + \tfrac{1}{2}\bar{G}t^2 + \tfrac{1}{6}\bar{N}t^3 + \cdots, \end{aligned} \tag{6}$$

where t takes the place of $X - x$ in (2), and where the bars indicate that the substitution (4), (5) has been carried out, so that, for example,

$$\bar{F}(x, y) \equiv F(x, y, f, P, Q). \tag{6'}$$

The coefficients of the powers of t in (6) are thus functions of the two parameters x, y.

The congruence (6) defines a field of lineal elements, which, in terms of the three parameters t, x, y, is given by

$$\begin{aligned} X &= x + t, \\ Y &= y + Pt + \tfrac{1}{2}\bar{F}t^2 + \tfrac{1}{6}\bar{M}t^3 + \cdots, \\ Z &= f + Qt + \tfrac{1}{2}\bar{G}t^2 + \tfrac{1}{6}\bar{N}t^3 + \cdots, \\ Y' &= P + \bar{F}t + \tfrac{1}{2}\bar{M}t^2 + \cdots, \\ Z' &= Q + \bar{G}t + \tfrac{1}{2}\bar{N}t^2 + \cdots. \end{aligned} \tag{7}$$

The condition for a normal congruence is *

$$(Y'XY) - (Z'ZX) + Y'(Z'YZ) - Z'(Y'YZ) = 0, \tag{8}$$

*In the present case $\bar{X} : \bar{Y} : \bar{Z} = 1 : Y' : Z'$. We may also use the convenient form due to Beltrami. Cf. BIANCHI-LUKAT, *Differentialgeometrie*, p. 340.

where the parentheses denote jacobians taken with respect to t, x, y. For example:

$$(Y'XY)=\begin{vmatrix} \bar{F}+\bar{M}t+\cdots & P_x+\bar{F}_x t+\cdots & P_y+\bar{F}_y t+\cdots \\ 1 & 1 & 0 \\ P+\bar{F}t+\cdots & P_x t+\cdots & 1+P_y t+\cdots \end{vmatrix}.$$

Expanding our results in powers of t, and writing down only the terms of first degree, we find

$$\begin{aligned}
&(Y'XY)=\bar{F}-PP_y-P_x+t\{\bar{M}-\bar{F}_x-P\bar{F}_y\}+\cdots,\\
&(Z'ZX)=Q_x f_y-Q_y f_x+QQ_y-Gf_y+t\{-\bar{N}f_y+\bar{G}_x f_y-\bar{G}_y f_x+Q\bar{G}_y\}+\cdots,\\
&(Z'YZ)=QQ_x+P(Q_y f_x-Q_x f_y)-\bar{G}f_x+t\{-\bar{N}f_x+\bar{F}(Q_y f_x-Q_x f_y)\\
&\qquad +\bar{G}(P_x f_y-P_y f_x)+QG_x+P(G_y f_x-G_x f_y)+Q(Q_x P_y-P_x Q_y)\}+\cdots,\\
&(Y'YZ)=QP_x+PP_y f_x-PP_x f_y-\bar{F}f_x\\
&\qquad +t\{-\bar{M}f_x+Q\bar{F}_x-\bar{F}Q_x+\bar{G}P_x+P(\bar{F}_y f_x-\bar{F}_x f_y+Q_x P_y-P_x Q_y)\}+\cdots.
\end{aligned} \tag{8'}$$

Substituting these values in (8), we find that the part free from t is

$$\frac{1}{Q}(1+P^2+Q^2)(PQ_x-QP_x-Q_y), \tag{9}$$

which vanishes in consequence of (5″). This is as it should be, since our ∞^2 curves are orthogonal to Σ by construction.

The terms containing the first power of t give

$$\begin{aligned}
&(1+P^2+Q^2)(P\bar{G}_x-Q\bar{F}_x-\bar{G}_y)\\
&\qquad +2\bar{F}\{(P^2+Q^2)(Q_x-PQ_y)+2\bar{G}fPP_y-(P^2+Q^2)P_x\}=0.
\end{aligned} \tag{10}$$

From (6′) we find

$$\bar{F}_x=F_x+F_z f'_x+F_p P_x+F_q Q_x,$$

with corresponding results for $\bar{G}_x$ and $\bar{G}_y$. Substituting these values, and observing from (5) and (5′) that

$$f_x=-\frac{1}{Q}, \qquad f_y=-\frac{P}{Q}, \qquad Q_y=PQ_x-QP_x,$$

we may reduce (10) to

$$\begin{aligned}
&2F\{PQP_x+Q^2Q_x\}+2G\{PP_y-(P^2+Q^2)P_x\}-(1+P^2+Q^2)\times\\
&\{QF_x-F_z-PG_x+G_y+(QF_p-QG_q-PG_p)P_x+G_pP_y+QF_qQ_x\}=0.
\end{aligned} \tag{10'}$$

This is then a necessary condition in order that the ∞^2 curves belonging to the quadruply infinite system (1) *and orthogonal to the surface* (4) *shall form a normal congruence.* The result is to hold in virtue of (4) and (5).

It is of course not a sufficient condition. It merely expresses the fact that the curves orthogonal to Σ are also orthogonal to some consecutive surface, that is, that the congruence is approximately normal to the first degree.

Our main problem is to find all systems (1) which have the orthogonality property with respect to *every* base surface Σ. It is then necessary that (10′) should be true for an arbitrary function $f(x, y)$. The function can be so selected that for any chosen values of x and y the quantities, f, P, Q, P_x, P_y, Q_x, shall take on arbitrary numerical values; for the only relation to be fulfilled is (5″) and this merely determines P_y. The condition (10′) must therefore hold identically. Arranging it in the form

$$(10'') \qquad (1 + P^2 + Q^2)C_0 + QC_1Q_x + C_2P_y - C_3P_x = 0,$$

and equating coefficients to zero, we find

$$\begin{aligned} &C_0 \equiv qF_x - F_z - pG_x + G_y = 0, \\ &C_1 \equiv (1 + p^2 + q^2)F_q - 2qF = 0, \\ &C_2 \equiv (1 + p^2 + q^2)G_p - 2pG = 0, \\ &C_3 \equiv (1 + p^2 + q^2)\{qG_q + pG_p - qF_p\} + 2pqF - 2(p^2 + q^2)G = 0. \end{aligned} \tag{11}$$

Integration of the second and third of these partial differential equations gives

$$F = f_1(p, x, y, z)(1 + p^2 + q^2), \qquad G = g_1(q, x, y, z)(1 + p^2 + q^2),$$

where f_1 and g_1 denote unknown functions of the four arguments indicated. Substituting these values in the fourth equation, we find

$$f_{1p} = g_{1q},$$

and therefore

$$f_1 = \psi - p\phi, \qquad g_1 = \chi - q\phi,$$

where ϕ, ψ, χ are functions of x, y, z only. The general solution of the last three equations of the set (11) is therefore

$$(12) \quad F = (\psi - p\phi)(1 + p^2 + q^2), \qquad G = (\chi - q\phi)(1 + p^2 + q^2).$$

We have still to satisfy the first equation of (11), which now reduces to

$$(13) \qquad \psi_z - \chi_y + p(\chi_x - \phi_z) + q(\phi_y - \psi_x) = 0.$$

The functions ϕ, ψ, χ must therefore satisfy the equations

$$(13') \qquad \psi_z - \chi_y = 0, \qquad \chi_x - \phi_z = 0, \qquad \phi_y - \psi_x = 0,$$

and hence are expressible as the derivatives of a common function in the form

$$(13'') \qquad \phi = L_x, \qquad \psi = L_y, \qquad \chi = L_z.$$

The solutions of the set (11) are therefore

$$(14) \quad F = (L_y - pL_x)(1 + p^2 + q^2), \qquad G = (L_z - pL_x)(1 + p^2 + q^2),$$

involving an arbitrary function L of x, y, z. The resulting system (1) is thus recognized to be a natural family.* This gives our fundamental

THEOREM I. *If a quadruply infinite system of curves in space, one passing in each direction through each point, has the property that those ∞^2 members of it which are orthogonal to an arbitrarily selected surface always form a normal congruence (that is, admit ∞^1 orthogonal surfaces), then the system must be a natural family.*

That natural families of trajectories actually have this property is known from Thomson and Tait's theorem. In the above discussion use has been made not of the complete condition for a normal congruence, but only of condition (10′) derived from the terms of the first order in t. We may therefore state a stronger converse result as follows:

THEOREM II. *The only systems of ∞^4 curves which have the property that the curves orthogonal to any surface are always orthogonal to some infinitesimally neighboring surface are those of the natural type.*

If a congruence of curves meets two neighboring surfaces orthogonally it need not meet ∞^1 surfaces orthogonally and therefore it approximates but need not coincide with a normal congruence. The above theorem shows however that if the weak requirement of approximate normal character be imposed on *all* the congruences obtained from the given quadruply infinite system, they will *all* be exactly normal.

We may further strengthen our theorem by demanding the orthogonality property for some instead of all surfaces. Our fundamental equations (11) resulted from the fact that $x, y, z, f, P, Q, P_x, P_y, Q_x$, might receive arbitrary numerical values. It will therefore be sufficient to take a manifold of surfaces sufficiently large to leave these quantities, or the equivalent quantities

$$(15) \qquad x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy},$$

unrestricted. Since these quantities define a differential surface element of the second order, we may state the result as follows:

*See these Transactions, vol. 10 (1909), p. 204.

THEOREM III. *If in theorem II, and, a fortiori, in theorem I, we consider instead of all surfaces, a manifold of surfaces containing all the ∞^8 differential elements of second order* (15), *the results are still valid.*

The smallest manifolds of this sort are composed of ∞^6 surfaces. The totality of quadrics would be more than sufficient. The paraboloids

$$z = a + bx + cy + a_1x^2 + b_1xy + c_1y^2$$

would be sufficient. In the discussion of base curves and points that follows, smaller manifolds of bases will present themselves.

2. CASE WHERE THE BASE IS A CURVE.

We now consider those curves of a given system

$$y'' = F(x, y, z, y', z'), \qquad z'' = G(x, y, z, y', z') \tag{1}$$

which meet an arbitrary curve

$$y = f(x), \qquad z = g(x) \tag{16}$$

orthogonally, and impose the condition for a normal congruence (to first approximation).

The ∞^2 possible initial elements (x, y, z, p, q) may be written in terms of two parameters x and p as follows:

$$y = f(x), \quad z = g(x), \quad q = Rp + S,$$

where

$$R \equiv -\frac{f'}{g'}, \qquad S \equiv -\frac{1}{g'}. \tag{17}$$

This results from the orthogonality of the directions $(1 : p : q)$ and $(1 : f' : g')$, expressed by

$$1 + pf' + qg' = 0, \tag{18'}$$

and giving

$$q = S + pR. \tag{18}$$

The congruence orthogonal to the base curve (16) may be written as a direction field in the form

$$\begin{aligned} X &= x + t, \\ Y &= f + pt + \tfrac{1}{2}\bar{F}t^2 + \cdots, \\ Z &= g + qt + \tfrac{1}{2}\bar{G}t^2 + \cdots, \\ Y' &= p + \bar{F}t + \tfrac{1}{2}\bar{M}t^2 + \cdots, \\ Z' &= S + pR + \bar{G}t + \tfrac{1}{2}\bar{N}t^2 + \cdots, \end{aligned} \tag{19}$$

where the parameters are t, x, and p. The bars indicate that the substitution (16), (18) has been effected, so that, for example,

$$\bar{F}(x, p) = F(x, f, g, p, S + pR). \tag{19'}$$

The requisite jacobians taken with respect to t, x, p are

$$(Y'XY) = f' - p + t(f' - p)\bar{F}_p + \cdots,$$

$$(Z'ZX) = R(S + pR - g') + t(S + pR - g')\bar{G}_p + \cdots,$$

$$(Z'YZ) = R\{pg' - f'(S + pR)\} + t[(S + pR)(S' + pR') + gR\bar{F} - g'\bar{G} + \{pg' - f'(S + pR)\}\bar{G}_p] + \cdots,$$

$$(Y'YZ) = pg' - f'(S + pR) + t[p(S' + pR') + f'R\bar{F} - f'\bar{G} + \{pg' - f'(S + pR)\}\bar{F}_p] + \cdots.$$

Developing the condition for a normal congruence in powers of t, we find that the absolute term vanishes identically, while the coefficient Ω_1 of the first power gives

$$\begin{aligned}(R - pS)\bar{F}_p - \{1 + pS(S + pR)\}G_p - 2R^2(S + pR)\bar{F} + pG - 2R\bar{F} \\ + R(S + pR)\bar{G} - \{p + R(S + pR)\}\{p\bar{G}_p - (S + pR)\bar{F}_p - G\} = 0.\end{aligned} \tag{20}$$

Eliminating S by means of (18), and substituting the values

$$\bar{F}_p = F_p + RF_q, \qquad \bar{G}_p = G_p + RG_q,$$

derived from (19'), we may reduce (20) to

$$\begin{aligned}2pG = (1 + p^2 + q^2)G_p + R[(1 + p^2 + q^2)(F_p - G_q) - 2pF + 2qG] \\ + R^2[(1 + p^2 + q^2)F_q - 2qF] = 0.\end{aligned} \tag{20'}$$

If this is to hold for all curves (16) it must be an identity. For the curve (16) can be so selected that x, y, z, p, q, R receive arbitrarily assigned numerical values. Hence we have the three partial differential equations

$$\begin{aligned}(1 + p^2 + q^2)F_q - 2qF = 0, \\ (1 + p^2 + q^2)G_p - 2pG = 0, \\ (1 + p^2 + q^2)(F_p - G_q) - 2pF + 2qG = 0.\end{aligned} \tag{20''}$$

This set is seen to be equivalent to the last three of the equations (11). The general solution is

$$F = (\psi - p\phi)(1 + p^2 + q^2), \qquad G = (\chi - p\phi)(1 + p^2 + q^2),$$

involving three arbitrary functions of x, y, z. The quadruply infinite system (1) thus takes the form

$$(21)\quad y'' = (\psi - y'\phi)(1 + y'^2 + z'^2), \qquad z'' = (\chi - z'\phi)(1 + y'^2 + z'^2).$$

This is recognized to be a *velocity system*, according to the terminology of the author's earlier paper. It is characterized geometrically by the property that the circles of curvature constructed at any point for those curves of the system which pass through that point form a bundle, i. e., have another point in common (property A). We have made use of merely one term in the complete condition for a normal congruence. Hence our result is

THEOREM IV. *Every velocity system has the property that those members of it which meet an arbitrarily selected base curve* orthogonally are also orthogonal to some neighboring surface (necessarily tubular). Conversely, if a quadruply infinite system has this property for all base curves it will be of the velocity type.*

The converse result remains valid if instead of all base curves we take a manifold which allows x, f, g, f', g' to take arbitrary values. The smallest number of base curves that would be sufficient is ∞^4. Two adequate manifolds of curves are made use of in the following

THEOREM V. *The converse result of theorem* IV *remains valid if the only bases employed are the ∞^4 straight lines of space; or, the ∞^4 curves of the system itself.*

§4. CASE WHERE THE BASE IS A POINT.

When the base is a point orthogonality means merely incidence. Consider then the ∞^2 curves of the system (1) which pass through a given point (x, y, z). Define the initial direction by $y' = p$, $z' = q$, where p and q are now arbitrary; the resulting congruence takes the form

$$(22)\quad \begin{aligned} X &= x + t, \\ Y &= y + pt + \tfrac{1}{2}Ft^2 + \tfrac{1}{6}Mt^3 + \cdots, \\ Z &= z + qt + \tfrac{1}{2}Gt^2 + \tfrac{1}{6}Nt^3 + \cdots, \\ Y' &= p + Ft + \tfrac{1}{2}Mt^2 + \cdots, \\ Z' &= q + Gt + \tfrac{1}{2}Nt^2 + \cdots. \end{aligned}$$

Our jacobians, calculated to terms of the third order in t, instead of merely

* This curve may or may not belong to the given system.

the first order as hitherto, are found to be

$$(Y'XY) = \tfrac{1}{2}F_q t^2 + \tfrac{1}{3}M_q t^3 + \cdots,$$

$$(Z'ZY) = \tfrac{1}{2}G_p t^2 + \tfrac{1}{3}N_p t^3 + \cdots,$$

$$-(Z'YZ) = qt + \tfrac{1}{2}\{pG_p + qF_p + 2qG_q\}t^2 + \tfrac{1}{2}\{q(F_pG_q - F_qG_p) - G_1 + qN_q + \tfrac{2}{3}pN_p + \tfrac{1}{3}qM_p\}t^3 + \cdots,$$

$$-(Y'YZ) = pt + \tfrac{1}{2}\{pG_q + qF_q + 2pF_p\}t^2 + \tfrac{1}{2}\{p(F_pG_q - F_qG_p) - F_1 + pM_p + \tfrac{2}{3}qM_q + \tfrac{1}{3}pN_q\}t^3 + \cdots,$$

where

$$F_1 \equiv F_x + pF_y + qF_z, \qquad G_1 \equiv G_x + pG_y + qG_z. \tag{22'}$$

Expanding the condition for a normal congruence, we find that there are no terms independent of t, the coefficient Ω_1 of t vanishes identically, the coefficient Ω_2 of t^2 gives

$$(1+q^2)F_q - (1+p^2)G_p + pq(F_p - G_q) + 2(pG - qF) = 0, \tag{23}$$

and the coefficient Ω_3 of t^3 gives

$$(1+q^2)M_q - (1+p^2)N_p + pq(M_p - N_q) + 3(pG - qF)(F_p + G_q) + 3p(G_x + pG_y + qG_z) - 3q(F_x + pF_y + qF_z) = 0. \tag{24'}$$

By means of (23) and (22') the equation (24') may be written in the more convenient form

$$(1+q^2)F_{1q} - (1+p^2)G_{1p} + pq(F_{1p} - G_{1q}) + 3(pF_1 - qG_1) = 0. \tag{24}$$

In order that a quadruply infinite system

$$y'' = F(x, y, z, y', z'), \qquad z'' = G(x, y, z, y', z') \tag{1}$$

shall have the property that the ∞^2 curves through an arbitrary point form a normal congruence, it is necessary that the functions $F(x, y, z, p, q)$ and $G(x, y, z, p, q)$ fulfill the conditions (23) *and* (24).

As stated above, the coefficient of the first power of t vanishes identically; in fact, if we think of the point as a small sphere, the curves passing through it, for any system (1), may be considered as meeting the sphere twice orthogonally. Equation (23) is the condition for approximation of the second degree; while (24) and (23) are the conditions for approximation of the third degree.

It is easily verified that all velocity systems

$$y'' = (\psi - y'\phi)(1 + y'^2 + z'^2), \qquad z'' = (\chi - z'\phi)(1 + y'^2 + z'^2),$$

satisfy equation (23). Substituting in (24), however, we find a restriction, which reduces to

$$\psi_x - \chi_y + p(\chi_x - \phi_x) + q(\phi_y - \psi_x) = 0.$$

Since this is to hold identically, the result is a natural family. In this case we know that the congruences are actually of the normal type. Hence

THEOREM VI. *The only velocity systems which have the property that the curves through an arbitrary point form a normal * congruence are the natural families.*

By combining this with the first part of theorem V, we obtain the following

THEOREM VII. *If a quadruply infinite system is such that the congruences constructed by taking as base every straight line and every point of space are normal, then the same will be true when the base is any surface or any curve, and the system must be natural.* †

It is easily seen that conditions (23) and (24), expressing the approximate fulfillment of the condition for a normal congruence in the case of base points, are not of themselves sufficient to characterize natural families. In fact the general solution of these two partial equations for the determination of F and G, one of the first and the other of the second order, would involve arbitrary functions of four arguments, and thus would include the natural type, which involves a function of only three arguments, as a very special case.

This question now arises: Can a system of ∞^4 curves, other than the known natural type, be such that the ∞^2 curves through an arbitrary point form a normal congruence? That the answer is in the affirmative we show by an example. The direct attack on the problem, according to the method hitherto pursued in this paper, would lead to great difficulties; it would require the discussion of the endless set of equations, of successively higher order, obtained by adjoining to (23) and (24) the conditions derived from the powers of t beyond the third.

To obtain an appropriate example, we consider the systems whose equations are of the special form

$$y'' = F(p, q), \qquad z'' = G(p, q), \tag{25}$$

the variables x, y, z being absent. Excluding the case where F and G both vanish, such systems are not natural. They are characterized by the fact that they admit the group of all translations. It will therefore be sufficient to test the character of the curves through a single point, for instance, the origin.

For the functions appearing in (25), we observe, from (22′), that the derived functions F_1 and G_1 vanish and hence condition (24) is satisfied. Condition

* It is sufficient to demand the third degree of approximation.

† It is sufficient to demand the first approximation in the case of the straight lines, and the third in the case of the points.

(23) may be fulfilled by taking either F or G at random and calculating the other function; this can easily be effected by quadratures. Thus all systems allowing the complete group of translations and such that the curves through an arbitrary point fulfill the normal condition up to terms of the third order may be obtained by quadratures.

It will be sufficient to consider the special case where G is assumed to vanish, that is, where the curves are assumed to lie in planes perpendicular to the xz plane. The systems are then found to be of the form

$$y'' = p^2 f\left(\frac{1+q^2}{p^2}\right), \qquad z'' = 0, \tag{26}$$

where f is an arbitrary function of one argument.

Consider now the particular system

$$y'' = p^2, \qquad z'' = 0, \tag{27}$$

included in this form. The integral curves are

$$y = \log \frac{a_1}{bx + a}, \qquad z = bx + b_1,$$

involving four arbitrary constants. The ∞^2 curves passing through the origin are

$$y = \log \frac{a}{x + a}, \qquad z = bx.$$

The differential equations of this congruence are

$$\frac{dx}{x} = \frac{dy}{e^y - 1} = \frac{dz}{z};$$

the corresponding total equation,

$$x\,dx + (e^y - 1)\,dy + z\,dz = 0,$$

is actually integrable, giving as the ∞^1 orthogonal surfaces

$$x^2 + z^2 + 2(e^y - y) = \text{constant}.$$

Our example (27), which evidently is not of the natural type, enables us to state

THEOREM VIII. *A quadruply infinite system may have the property that the curves through an arbitrary point form a normal congruence, without necessarily being a natural system.*

Not all systems of the form (26) have the required property. For consider the particular system

$$y'' = \tfrac{1}{2}(1 + q^2), \qquad z'' = 0, \tag{28}$$

whose integral curves are the parabolas

$$y = (1 + b^2)x^2 + ax + a_1, \qquad z = bx + b_1.$$

The congruence through the origin is

$$y = (1 + b^2)x^2 + ax, \qquad z = bx,$$

with the differential equations

$$\frac{dx}{x} = \frac{dy}{y + x^2 + z^2} = \frac{dz}{z}.$$

Since the corresponding total equation

$$xdx + zdz + (y + x^2 + z^2)dy = 0$$

is not integrable, the congruence is not normal.

This example shows that a system may be such that the curves through an arbitrary point form a congruence which is normal to the third degree of approximation without being exactly normal.

§ 5. The family of surfaces related to an arbitrary system.

Given an arbitrary quadruply infinite system of curves

$$y'' = F(x, y, z, p, q), \qquad z'' = G(x, y, z, p, q), \tag{29}$$

where p and q stand for y' and z' respectively, it is evident that in general there will exist no surface such that the ∞^2 curves of the system which meet the surface orthogonally form a normal congruence. If, however, it is merely required that the ∞^2 curves shall be of the normal character to the first approximation, that is, shall be orthogonal to some consecutive surface, then solutions will always exist.

The requisite condition on the surface

$$z = f(x, y) \tag{30}$$

is furnished by equation (10′). In the present discussion F and G are considered as the given functions and f is to be found. Rearranging the equation in terms of second derivatives of f, we write it

$$B_0 + B_1 f_{xx} + B_2 f_{xy} + B_3 f_{yy} = 0, \tag{31}$$

where the coefficients are functions of x, y, z, p, q, determined by F and G as follows:

$$\begin{aligned}
qB_0 &\equiv (1 + p^2 + q^2)(qF_x - F_z - pG_x + G_y), \\
B_1 &\equiv (1 + p^2 + q^2)(pqF_p + q^2F_q - p^2G_p - pqG_q) + 2(p^2 + q^2)(pG - qF), \\
B_2 &\equiv (1 + p^2 + q^2)(2pG_p + qG_q - qF_p) + 2pqF - 2(q^2 + 2p^2)G, \\
B_3 &\equiv 2pG - (1 + p^2 + q^2)G_p.
\end{aligned} \tag{31′}$$

Here p and q take the place of P and Q respectively in the discussion of § 2, so that we must substitute

$$p = \frac{f_y}{f_x}, \qquad q = -\frac{1}{f_x}. \tag{32}$$

After the substitution (30), (32), the coefficients B_i become known functions of x, y, f, f_x, f_y. Denoting these new functions by $\bar{B}_i$, we have

THEOREM IX. *With any quadruply infinite system of curves* (29), *there is associated a certain family of surfaces, namely, the solutions of the Monge partial differential equation of the second order*

$$\bar{B}_0 + \bar{B}_1 f_{xx} + \bar{B}_2 f_{xy} + \bar{B}_3 f_{yy} = 0, \tag{33}$$

where the coefficients are found from F and G by differentiations and eliminations. The ∞^2 curves of the given system which are orthogonal to any surface of this family are also orthogonal to some consecutive surface, that is, form an approximate normal congruence.

In case the system is of the natural type, equation (33) is of course an identity in f. In fact the set of equations obtained by putting the four coefficients $\bar{B}_i$, or the corresponding coefficients B_i, separately equal to zero is equivalent to the set (11) whose solution yielded the natural systems in the discussion of § 2.

When will (33) reduce to an equation of the first order? The conditions for this are

$$B_1 = B_2 = B_3 = 0,$$

which, in the notation of equations (11), are seen to be equivalent to

$$C_1 = C_2 = C_3 = 0.$$

The system must therefore be of the velocity type

$$y'' = (\psi - p\phi)(1 + p^2 + q^2), \qquad z'' = (\chi - q\phi)(1 + p^2 + q^2).$$

In this case, (33) reduces to

$$(\psi_z - \chi_y) f_x + (\chi_x - \phi_z) f_y - (\phi_y - \psi_x) = 0. \tag{34}$$

The integral surfaces are generated by the curves (characteristics)

$$\frac{dx}{\psi_z - \chi_y} = \frac{dy}{\chi_x - \phi_z} = \frac{dz}{\phi_y - \psi_x}. \tag{35}$$

These are the curl lines connected with the vector field* ϕ, ψ, χ.

THEOREM X. *For a general system of curves the associated equation* (33) *is actually of the second order, the only exceptions arising as follows: for*

* This represents the force which generates the velocity system.

a natural system, the equation holds identically; for any other velocity system the equation is merely of the first order. In the first case the associated family of surfaces is thus larger, and in the second smaller, than usual.

This deficiency refers to proper surfaces. We have already seen that, in the velocity type, an arbitrary *curve* employed as base gives rise to a congruence of approximately normal character. Thus, in addition to the proper surfaces (34), all curves may be regarded as limiting cases belonging to our family (33). In general we may with LIE replace the concept surface by the more general concept of union (*Elementverein*), that is, manifold of ∞^2 surface elements in united position. We might in fact have started out from this point of view and thereby should have avoided the separate discussion of curves and points. However this would have required the introduction of appropriate systems of homogeneous coördinates for differential surface elements.

One of our former results may be stated in the form: If the equation (33) is satisfied by all quadrics, or merely by the paraboloids

$$z = a + bx + cy + a_1x^2 + b_1xy + c_1y^2,$$

it must hold identically, so that the corresponding system must be natural. We now discuss the analogous question for spheres.

The ∞^4 spheres of space may be defined by the set of partial differential equations

$$\frac{f_{xx}}{1+f_x^2} = \frac{f_{xy}}{f_xf_y} = \frac{f_{yy}}{1+f_y^2}. \tag{36}$$

If then the associated family of surfaces is to include the spheres, equation (33) must be satisfied in consequence of (36). Eliminating f_{xx} and f_{yy} by means of (36), and making use of (32), we find that (33) reduces to

$$pB_0 + \{(1+q^2)B_1 + pB_2 + (p^2+q^2)B_3\}f_{xy} = 0.$$

This must be true for all values of f_{xy}; hence

$$B_0 = 0, \qquad (1+q^2)B_1 + pB_2 + (p^2+q^2)B_3 = 0,$$

that is,

$$qF_x - F_z - pG_x + G_y = 0, \tag{37}$$

$$(1+q^2)F_q - (1+p^2)G_p + pq(F_p - G_q) + 2(pG - qF) = 0. \tag{38}$$

The latter is recognized as the condition that all points shall give rise to approximately (of first degree) normal congruences. The other equation, (37), is easily seen to be the condition that (33) shall be satisfied by all planes.

If for each point and for each plane as base, the congruence of orthogonal curves is approximately of normal character, then the same is true for an arbitrary sphere. The systems of this sort are defined by equations (37), (38).*

* The result involves arbitrary functions of four arguments, hence is more general than the natural type.

It is easily seen that the only velocity systems satisfying condition (37) are those of the natural type. Making use of theorem V, we thus have

THEOREM XI. *If a system of curves is such that the ∞^2 curves orthogonal to any straight line or to any plane are orthogonal to some neighboring surface then the system must be natural.*

If for an arbitrary curve as base the congruence of orthogonal curves is (exactly) normal, then the same must be true for any point as base. For the curve may be made to shrink up to the point. Hence, by combining the converse result of theorem IV with theorem VI, we have

THEOREM XII. *Natural systems are characterized by the requirement that the curves of the system which are orthogonal to an arbitrarily selected curve (inside or outside the system) form a normal congruence.*

It is even sufficient to take as base curves merely the ∞^6 circles of space.

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THE INTRODUCTION OF IDEAL ELEMENTS AND A NEW DEFINITION OF PROJECTIVE n-SPACE*

BY

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Introduction.

The ideal elements of projective geometry are usually introduced by means of the parallel and congruence axioms. The idea of defining the ideal elements without the assumption of the parallel axiom is due to KLEIN.† It was developed by PASCH;‡ by SCHUR;§ by BONOLA;‖ and by VEBLEN.¶ In all of these developments a three-dimensional geometry is assumed. The problem of defining the ideal elements in a plane geometry satisfying only order relations is closely connected with the problem of finding the necessary and sufficient condition that a plane may be a part of a three-space in which the axioms of order are satisfied. This condition is stated by HILBERT** to be the validity of the Desargues theorem in the plane. The Desargues theorem may be proved in a three-dimensional space satisfying only order relations, but can not be proved in the corresponding plane geometry, without additional assumptions, e. g., of the parallel and congruence axioms.

In this paper plane axioms of order will be assumed in the form given them by VEBLEN, the undefined elements being taken as the *point*, and a relation among points, called *order*. The first eight axioms are identical, except for notation, with his. Another axiom, one of closure, is then introduced, limiting the set of *points* considered to a plane. Two more axioms are then introduced, forms of the Desargues theorem, and of its converse, in terms of the set of points satisfying only order relations.

* Presented to the Society (Chicago) under a different title, April 22, 1905.

† *Ueber die sogenannte Nicht-Euklidische Geometrie*, Mathematische Annalen, vol. 6 (1872), p. 132.

‡ *Vorlesungen über neuere Geometrie*, pp. 40–72.

§ *Ueber die Einführung der sogenannten idealen Elemente in die projective Geometrie*, Mathematische Annalen, vol. 39 (1891), pp. 113–128.

‖ *Sulla introduzione degli enti improprii in Geometria projettiva*, Giornale di Matematiche, vol. 38 (1900), p. 105.

¶ *A system of axioms for geometry*, these Transactions, vol. 5 (1904), pp. 343–384.

** *Grundlagen der Geometrie, Festschrift* (1899), § 30.

On the basis of these eleven axioms, a new set of elements, called *pencils*, including in particular the pencils of lines centered at points, are defined, and their properties deduced. This set of pencils is shown to satisfy the incidence axioms and the general Desargues theorem of projective geometry, but not necessarily the Pascal theorem, well known * to be independent of the Desargues theorem, and even stronger assumptions. It is also shown that if we leave out of this set of pencils a subset called a *range*, the remaining elements satisfy the original axioms, I to XI.

In Section IV, a new set of elements called three-points, including in particular the projective points (i. e. pencils) of the plane, is defined, and this set is shown to form a projective three-space containing the plane set as a subset.

In Section V, an extension is made to n-space by generalizing the method of Section IV.

The notation used, though sometimes cumbersome, is very convenient in giving a concise expression for the proofs; and by displaying the configurational character of many of the figures used, it has been found of value in devising the proofs of the theorems. The *points* which are characterized by Axioms I–XI will be denoted by the symbol P_k, the P simply indicating that the element belongs to the class of points, and the subscript being a mark, not necessarily a single digit or letter, and the order of the digits or letters in the subscript immaterial, used to distinguish the various points; for instance, conveniently, the notation P_{xy}, $x, y = 1, \cdots, 5$, $x \neq y$, and $P_{xy} = P_{yx}$, will be used for the configuration of ten points called the Desargues configuration; the notation l_k will be used for lines, the k again being a mark of one or more digits or letters, the order of which is immaterial; the notation l_{123} will be used where convenient for a line containing points P_{12}, P_{13}, P_{23}. Such implications of the notation, while useful, are avoided in the statement of the proofs, as far as possible, but the notation is used consistently, even the axioms themselves being stated in it, in order to accustom the reader to it. Other forms of notation will be explained, as they occur.

The independence of the Axioms X and XI from Axioms I–IX may be shown from the Non-Desarguesian geometry of MOULTON † if the "line of break" of this geometry is a line of our points. The relation of Axioms X and XI on the basis of Axioms I–IX has not been determined.

This problem was first suggested to the author by Professor O. VEBLEN, to whom, and to Professor E. H. MOORE, he tenders his grateful acknowledgment for many helpful suggestions and criticisms.

* *Grundlagen der Geometrie, Festschrift*, § 31. The form of Pascal's theorem here referred to is, of course, that for the degenerate conic. (Theorem of Pappus.)

† *A simple Non-Desarguesian geometry*, Transactions of the American Mathematical Society, vol. 3 (1902), pp. 192–195.

The Axioms and Definitions.

AXIOM I. There exist at least two distinct points.

AXIOM II. If points P_1, P_2, P_3 are in the order $P_1P_2P_3$, they are in the order $P_3P_2P_1$.

AXIOM III. If points P_1, P_2, P_3 are in the order $P_1P_2P_3$, they are not in the order $P_2P_3P_1$.

AXIOM IV. If points P_1, P_2, P_3 are in the order $P_1P_2P_3$, then P_1 is distinct from P_3.

AXIOM V. If P_1 and P_2 are any two distinct points, there exists a point P_3, such that P_1, P_2, P_3 are in the order $P_1P_2P_3$.

If P_1 and P_2 stand for the same points, we write $P_1 = P_2$; if for distinct points, $P_1 \neq P_2$.

DEFINITION 1. The line P_1P_2, or conveniently, l_{12}, consists of P_1 and P_2 and all points P_k in any of the orders, $P_1P_kP_2$, $P_1P_2P_k$, $P_kP_1P_2$. The subscripts 1, 2 in the notation l_{12} refer to the points by which the line is determined.

The points P_k in the order $P_1P_kP_2$ constitute the *segment* P_1P_2.

AXIOM VI. If points P_3 and P_4 ($P_3 \neq P_4$) lie on the line l_{12} (i. e., P_1P_2) then P_1 lies on the line l_{34} (i. e., P_3P_4).

AXIOM VII. If there exist three distinct points, there exist three points P_1, P_2, P_3 not in any one of the orders $P_1P_2P_3$, $P_1P_3P_2$, or $P_3P_1P_2$.

DEFINITION 2. Points of the same line are *collinear*. Three distinct non-collinear points, P_1, P_2, P_3 are the *vertices* of a *triangle* $P_1P_2P_3 = T_{123}$, whose *sides* are the segments P_1P_2, P_1P_3, P_2P_3, and whose *boundary* consists of its vertices and the points on its sides. Two triangles having the same vertices are identical.

AXIOM VIII. (Triangle transversal axiom.) If three distinct points $P_1P_2P_3$ do not lie on the same line, and P_4 and P_5 are two points in the orders $P_2P_3P_4$ and $P_3P_5P_1$, respectively, then a point P_6 exists in the order $P_1P_6P_2$, and such that P_4, P_5, P_6 lie on the same line.

DEFINITION 3. If P_1, P_2, P_3 are three non-collinear points, the *plane* $P_1P_2P_3$ consists of all the points collinear with any two points of the sides of the triangle $P_1P_2P_3$.

AXIOM IX. The set of points P_k here considered constitutes a plane.

DEFINITION 4. A triangle is said *to lie on three lines* if it has a vertex on each of these lines.

DEFINITION 5. Three distinct lines l_{340}, l_{341}, l_{342} are *in pencil* if there exists on these lines a pair of triangles having no side common $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$, P_{30} and P_{40}, $P_{30} \neq P_{40}$, being points of the line l_{340}, etc., such that the lines $P_{30}P_{31}$ and $P_{40}P_{41}$ meet* in a point P_{01}, the lines $P_{31}P_{32}$ and

* Two lines *meet* only if distinct.

$P_{41}P_{42}$ meet in a point P_{12}, and the lines $P_{30}P_{32}$ and $P_{40}P_{42}$ meet in a point P_{02}, and such that the points P_{01}, P_{02}, P_{12} are collinear.

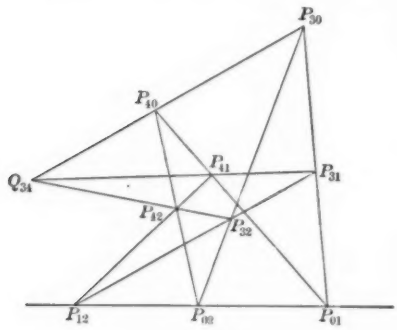

FIG. 1.

The figure of Definition 5 (see Fig. 1) will be called a Desargues configuration, and the notation

$$(1)\qquad \begin{array}{ccc} P_{12} & P_{02} & P_{01} \\ \hline P_{40} & P_{41} & P_{42} \\ P_{30} & P_{31} & P_{32} \\ & Q_{34} & \end{array}$$

will be used when by hypothesis the triads of points, $P_{12}P_{02}P_{01}$, $P_{12}P_{41}P_{42}$, $P_{12}P_{31}P_{32}$, $P_{02}P_{40}P_{42}$, $P_{02}P_{30}P_{32}$, $P_{01}P_{40}P_{41}$, $P_{01}P_{30}P_{31}$ are collinear, and the triads $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$ are vertices of triangles; by the above definition, then, the lines $P_{40}P_{30}$, $P_{41}P_{31}$, and $P_{42}P_{32}$ will be *in pencil*, Q_{34}.

DEFINITION 6. The set of all lines *in pencil* with a given pair, together with this pair, constitute a pencil, determined by the two lines, and by Theorem VI, determinable by *any* two lines in it. The notation Q_k will be used for pencils, the mark k, as before, being any convenient symbol of one, two, or more digits or letters, these usually referring to the lines by which it is determined, and the order of the digits of the subscript being immaterial as far as denoting the pencil is concerned. A line is said to *belong* to a pencil if it is one of the lines of the pencil.

AXIOM X. If l_{340}, l_{341}, l_{342} are three lines in a pencil, and $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$ are any two triangles on these lines, P_{30} and P_{40} lying on l_{340} (but not necessarily distinct), etc., such that the lines

$$\begin{array}{lll} P_{30}P_{31}=l_{301} & \text{and} & P_{40}P_{41}=l_{401} \text{ meet in a point } P_{01}, \\ P_{31}P_{32}=l_{312} & \text{and} & P_{41}P_{42}=l_{412} \text{ meet in a point } P_{12}, \\ P_{30}P_{32}=l_{302} & \text{and} & P_{40}P_{42}=l_{402} \text{ meet in a point } P_{02}, \end{array}$$

and P_{01}, P_{12}, P_{02} are distinct and determinate, then P_{01}, P_{12}, and P_{02} are collinear.

The figure of Axiom X (see Fig. 2) will be represented by the notation

$$(2)\qquad \begin{array}{ccc} & Q_{34} & \\ P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ \hline P_{12} & P_{02} & P_{01} \end{array}$$

which indicates that by hypothesis the lines $P_{30}P_{40}$, $P_{31}P_{41}$, $P_{32}P_{42}$ are in pencil Q_{34}, and the triads of points P_{30} $P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$ are the vertices of triangles on these lines, and the triads of points $P_{30}P_{31}P_{01}$, $P_{40}P_{41}P_{01}$, $P_{30}P_{32}P_{02}$, $P_{40}P_{42}P_{02}$, $P_{31}P_{32}P_{12}$, $P_{41}P_{42}P_{12}$, exist and are respectively collinear. Axiom X states, then, that the points $P_{12}P_{02}P_{01}$ are collinear.

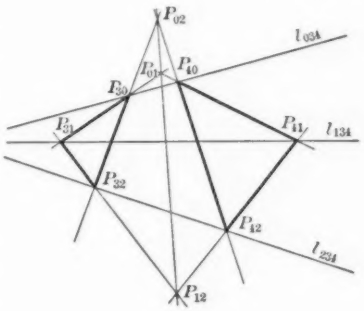

FIG. 2.

DEFINITION 7. Three distinct pencils Q_{01}, Q_{12}, Q_{02} are *in range*, if there exist two triangles, $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$, whose sides are all distinct and such that $P_{30}P_{31}$ and $P_{40}P_{41}$ belong to Q_{01}, $P_{31}P_{32}$ and $P_{41}P_{42}$ belong to Q_{12}, and $P_{30}P_{32}$ and $P_{40}P_{42}$ belong to Q_{02}; and such that the lines $P_{30}P_{40}$, $P_{31}P_{41}$, and $P_{32}P_{42}$ are in a pencil Q_{34}.

The notation used for the figure of Definition 7 will be identical with that of Axiom X, except that for P_{12}, P_{02}, P_{01} corresponding symbols Q_{12}, Q_{02}, Q_{01} will be used, thus:

$$\begin{array}{ccc} & Q_{34} & \\ P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ \hline Q_{12} & Q_{20} & Q_{01} \end{array}$$

It will be observed that whenever the hypothesis of Axiom X is fulfilled, so is the hypothesis in Definition 7, and to the points $P_{12}P_{02}P_{01}$ of Axiom X correspond pencils $Q_{12}Q_{02}Q_{01}$ which by Definition 7 are in range; for whether the pairs of lines $P_{30}P_{32}$ and $P_{40}P_{42}$, $P_{31}P_{32}$ and $P_{41}P_{42}$, and $P_{30}P_{31}$ and $P_{40}P_{41}$, determine points or not, they *do* determine *pencils* (cf. Corollary to Theorem VII).

AXIOM XI. If three pencils Q_{01}, Q_{02}, Q_{12} are in range, *every* pair of triangles $P_{30}P_{31}P_{32}$ and $P_{40}P_{41}P_{42}$, such that

lines $P_{30}P_{31}$ and $P_{40}P_{41}$ belong to Q_{01},
lines $P_{31}P_{32}$ and $P_{41}P_{42}$ belong to Q_{12},
lines $P_{30}P_{32}$ and $P_{40}P_{42}$ belong to Q_{02},

has the property that the lines $P_{30}P_{40}$, $P_{31}P_{41}$ and $P_{32}P_{42}$ are in a pencil Q_{34}.

The figure of Axiom XI will be denoted by the notation:

$$\begin{array}{ccc} Q_{12} & Q_{20} & Q_{01} \\ \hline P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ & Q_{34} & \end{array}$$

The inversion of the notational form is to call attention to the change in the hypothesis. That Definitions 6 and 7 and Axioms X and XI all involve the Desargues configuration explains the similarity of the notation.

DEFINITION 8. The set of all pencils in range with a given pair of ranges, together with this pair, constitutes a *range*. The notation r_k will be used for ranges, the mark k being again any convenient digit or letter, or combination of them in which the order is immaterial, so far as denoting the range is concerned.

From Axioms I–IX, such theorems as the following may be deduced (cf. VEBLEN, loc. cit.):

a) Between any two distinct points of a line there is another point (P_1 is between P_2 and P_3 if they are in the order $P_2P_1P_3$).

b) If a line, not a side of a triangle, meets the perimeter of the triangle in one point not a vertex, then it meets the perimeter in another distinct point.

c) To any four points of a line the notation P_1, P_2, P_3, P_4, can be so assigned that they are in the order, $P_1P_2P_3P_4$, i. e. in the orders, $P_1P_2P_3$, $P_1P_3P_4$, $P_1P_2P_4$, and $P_2P_3P_4$.

d) Any line divides the plane into two regions which may be denoted by + and −, and have the property that any segment joining a point of + to a point − contains a point of the line.

e) Two non-intersecting lines divide a plane into three regions, which can be conveniently denoted by ++, +−, −−.

f) Two intersecting lines divide a plane into four regions, which may be denoted by ++, +−, −+, −−.

g) Three lines forming an actual triangle divide the plane into seven regions, which may be denoted by +++, ++−, +−+, +−−, −−+, −++, −+−.

SECTION I.

Consequences of Axioms I–X.

Our main object in Section I is to show that a line is uniquely determined by a point and the fact that it is in pencil with two other given lines neither of which passes through the point. This is done without any use of Axiom XI.

THEOREM I. If the lines l_{034}, l_{134}, l_{234} are in pencil Q_{34}, and l_{034} and l_{134} meet in a point P_{30}, then l_{234} passes through P_{30}. (Fig. 3.)

Proof. Suppose the theorem untrue. Take P_{32}, any point of l_{234} (P_{32} not a point of l_{034} or l_{134}). Take on l_{134} points P_{71} and P_{61} each distinct from P_{30}, not on l_{234}, in the order $P_{71}P_{30}P_{61}$, and such there is no point of l_{234} on the segment $P_{61}P_{71}$. The line l_{034} cuts the side $P_{71}P_{61}$ of the triangle $P_{71}P_{61}P_{32}$. Hence it cuts either the segment $P_{32}P_{71}$ or the segment $P_{32}P_{61}$ in a point P_t. Call the side cut $P_{32}P_{31}$, so that either P_{71} or P_{61} is P_{31}, and call the other P_{81}. Choose a point P_{02} in the order $P_{32}P_{30}P_{02}$, and a point P_y in the order

$P_{31}P_{32}P_y$. The segment $P_{02}P_y$ contains a point of the segment $P_{32}P_{81}$ or does not. If it does, use P_y as P_{01} below. If not, take a point P_r on the segment $P_{32}P_{81}$, and the line $P_{02}P_r$ will meet the segment $P_{32}P_y$ in a point

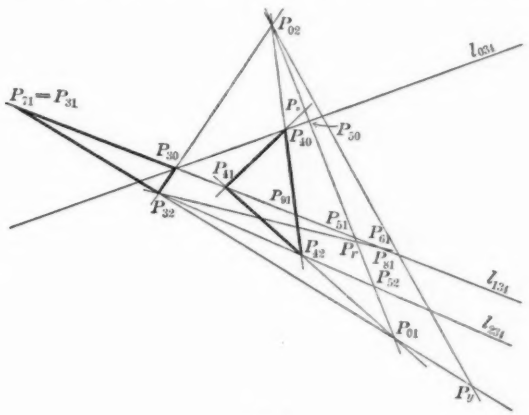

FIG. 3.

P_{01}, which is in the order $P_{02}P_rP_{01}$. Then the segment $P_{02}P_{01}$ will meet l_{034}, l_{134}, and l_{234} in points P_{50}, P_{51}, P_{52}, respectively, in the order $P_{02}P_{50}P_{51}P_rP_{52}P_{01}$. Take P_{41} in the order $P_{30}P_{41}P_{51}$. Then the segment $P_{41}P_{01}$ meets l_{234} in a point P_{42} of the segment $P_{32}P_{52}$. The segment $P_{42}P_{02}$ contains points, P_{40} on the line l_{034}, and P_{91} on the line l_{134}, in the order $P_{02}P_{40}P_{91}P_{42}$, and P_{91} is in the order $P_{30}P_{41}P_{91}P_{51}$. Therefore lines $P_{41}P_{40}$ and $P_{02}P_{01}$ determine a point P_s in the orders $P_{41}P_{40}P_s$ and $P_{02}P_sP_{50}$. We have then the Desargues configuration

$$\begin{array}{ccc} & Q_{34} & \\ P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ \hline P_{01} & P_{02} & P_{41}, \end{array}$$

in which the hypothesis of Axiom X is satisfied, but P_{01}, P_{02}, P_{41} are not collinear. Hence the assumption that l_{234} does not pass through P_{02} is contradictory to Axiom X. Hence the theorem is true.

THEOREM II. Any three lines l_{034}, l_{134}, l_{234} incident at a point P_{34} are in a pencil Q_{34}. (Fig. 4.)

Proof. Choose $P_{30} \neq P_{34}$ on l_{034} and $P_{61} \neq P_{34}$ on l_{134}, and $P_{32} \neq P_{34}$ on l_{234}, in the order $P_{32}P_{61}P_{30}$. Then choose P_{02} in the order $P_{32}P_{61}P_{30}P_{02}$; P_{31} in order $P_{34}P_{61}P_{31}$; and P_{12} in the order $P_{31}P_{32}P_{12}$, and such that segment $P_{32}P_{12}$ contains no point of line $P_{34}P_{02}$. Then, by consideration of

triangle $P_{02}P_{32}P_{12}$, $P_{02}P_{12}$ contains points P_{50}, P_{51}, in order $P_{02}P_{50}P_{51}P_{12}$, and P_{50} is on the segment $P_{34}P_{30}$, and P_{51} is on the segment $P_{34}P_{61}$. $P_{31}P_{30}$ meets $P_{02}P_{12}$ in a point P_{01}, in the order $P_{02}P_{01}P_{50}$. Take P_{41} on the segment $P_{34}P_{51}$. The segment $P_{41}P_{01}$ contains P_{40} on the segment $P_{34}P_{59}$. $P_{02}P_{40}$ meets $P_{41}P_{12}$ in a point P_{42}. Then $P_{32}P_{42}$ is a line in pencil with l_{034} and l_{134}. By Theorem I, the line $P_{32}P_{42}$ must pass through P_{34}, but this line contains the points P_{32} and P_{34}, and hence is the line l_{234}.

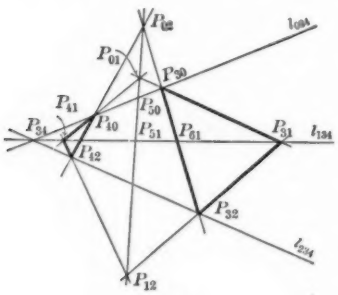

FIG. 4.

COROLLARY. If three lines are in pencil they meet in a point, or else no two of them have a point in common.

DEFINITION 8. A line l_k is *between* the non-intersecting lines l_m and l_n, if every segment P_mP_n (P_m on l_m, P_n on l_n) contains a point P_k of l_k.

DEFINITION 9. If three lines are in pencil and no two of them intersect, we will call the pencil *non-intersecting*.

THEOREM III. If l_{034}, l_{134}, l_{234} are three lines of a *non-intersecting* pencil, Q_{34}, then one of them is always *between* the other two.

LEMMA. If l_k, l_m, l_n are three non-intersecting lines, and if l_m cuts *one* segment joining points of l_k and l_n, then it cuts every such segment, and hence is between l_k and l_n. The proof follows readily from Axiom VIII and is omitted.

Proof of Theorem III. (Fig. 5.) Take P_{30} on l_{034}, P_{31} on l_{134}, and P_{32} on l_{234}. Then not more than one of the lines l_{034}, l_{134}, l_{234} enters the triangle $P_{30}P_{31}P_{32}$, since by hypothesis the pencil is non-intersecting. Then, if *one* does, it cuts the opposite side of the triangle and hence by the lemma it is between the other two lines. We will now prove that the case of all three of the lines l_{034}, l_{134}, l_{234} exterior to the triangle can not occur. Suppose that it did occur. Let the plane be divided by the sides of the triangle into seven regions as in Theorem g, p. 146.

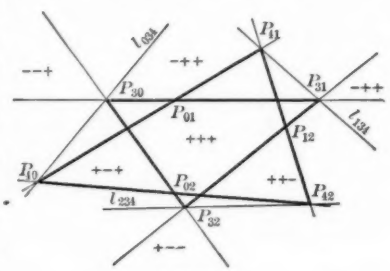

FIG. 5.

Choose P_{40} on l_{034} in $+-+$, and P_{41} on l_{134} in $-++$, and P_{42} on l_{234} in $++-$. Then $P_{40}P_{41}$ meets segment $P_{30}P_{31}$ in a point P_{01}, and $P_{41}P_{42}$ meets segment $P_{31}P_{32}$ in a point P_{12}, and $P_{42}P_{40}$ meets segment $P_{32}P_{30}$ in a point P_{02} and by hypothesis we have (2); but the points P_{12}, P_{02}, P_{01} are not collinear since they are on distinct sides of the triangle. Hence our supposition that no one of the lines enters the triangle is impossible.

DEFINITION 10. If three lines of a pencil are incident at a point, then their common point is the *center* of the pencil.

THEOREM IV. If l_{034}, l_{134}, and l_{234} are three lines in pencil, and such that, if the pencil be non-intersecting,* l_{134} is between l_{034} and l_{234}, and l_{934} ($l_{934} \neq l_{234}$) is a line meeting l_{234} in a point P_{32} (P_{32} not on l_{034} and not on l_{134}), then l_{034}, l_{134}, and l_{934} are not in pencil.

Proof. (*A*) If the pencil has a center, the theorem follows from Theorem I.

(*B*) Suppose the pencil has no center. (Fig. 6.)

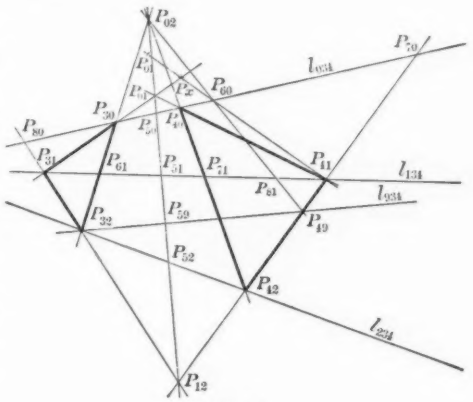

FIG. 6.

Suppose there were a line l_{934} satisfying the conditions of the hypothesis. Then l_{934} can not meet either l_{034} or l_{134}, by Theorem I, and also l_{134} will be between l_{034} and l_{934}. Let P_{80} and P_{70} be any two distinct points on l_{034}. Let P_{12} be any point in the order $P_{80}P_{32}P_{12}$. The segment $P_{80}P_{32}$ meets l_{134} in a point P_{31}. Since neither l_{234} nor l_{934} meets either l_{034} or l_{134}, the line $P_{70}P_{12}$ meets l_{134} in a point P_{41}, l_{234} in a point P_{42}, and l_{934} in a point P_{49}, and these points are in the order $P_{70}P_{41}P_{42}P_{12}$ and $P_{70}P_{41}P_{49}P_{12}$. Take P_{50} on the segment $P_{80}P_{70}$. Then take P_{02} in the order $P_{12}P_{50}P_{02}$ and the line $P_{12}P_{50}$ will meet l_{234} in a point P_{52}, l_{934} in a point P_{59}, and l_{134} in a point P_{51}, and these points are in the orders $P_{12}P_{52}P_{51}P_{50}P_{02}$ and $P_{12}P_{59}P_{51}P_{50}P_{02}$. $P_{02}P_{32}$ contains points P_{30} on l_{034} and P_{61} on l_{134}, in the order $P_{02}P_{30}P_{61}P_{32}$. $P_{02}P_{42}$ meets l_{034} in P_{40} and l_{134} in P_{71}, in the order $P_{02}P_{40}P_{71}P_{42}$. The line $P_{31}P_{30}$ meets $P_{52}P_{50}$ in a point P_{01}, in the orders $P_{31}P_{30}P_{01}$ and $P_{02}P_{01}P_{50}$. $P_{41}P_{40}$ also meets the line $P_{12}P_{02}$ in a point which by Axiom X is the same as P_{01}, from the Desargues configuration (2).

Now the line $P_{02}P_{49}$ meets l_{134} in P_{81}, and l_{034} in P_{60}, and these points are in the order $P_{02}P_{60}P_{81}P_{49}$. The line $P_{41}P_{60}$ meets the segment $P_{02}P_{50}$ in P'_{01}. But P'_{01} can not be P_{01}, since P_{40} and P_{60} are distinct. Also $P_{41}P_{60}$ and $P_{31}P_{30}$, since each enters the triangle $P_{02}P_{30}P_{60}$ at a vertex, meet in a point P_x. But this point is not collinear with P_{12} and P_{02}, since $P_{31}P_{30}$ and $P_{41}P_{40}$ meet $P_{12}P_{02}$ in distinct points, P_{01} and P'_{01}. Then we should have the Desargues configuration

* Note that the proof of the theorem as given would not be valid except for this restriction. The result is proved in general in Theorems VII and VIII.

$$\begin{array}{ccc} & Q_{34} & \\ P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ \hline P_{12} & P_{02} & P_x \end{array}$$

In this configuration, if l_{034}, l_{134}, and l_{934} are in pencil Q_{34}, as we have supposed, then by Axiom X the points P_{12}, P_{02}, P_x must be collinear. But we have proved that they are not collinear, and hence the lines l_{034}, l_{134}, and l_{934} are not in pencil.

THEOREM V. If l_{045}, l_{145}, and l_{245} are three lines in a non-intersecting pencil, and l_{045}, l_{145}, and l_{345} ($l_{245} \neq l_{345}$) are three lines in a non-intersecting pencil, then two of these lines are always between the other two.

Proof. Since no two of the lines meet, it is only necessary to find a single segment joining points of two of them, and containing points of the remaining two, as the lemma to Theorem III shows.

Let the regions into which the plane is divided by the lines l_{045} and l_{145} be denoted by $++$; $-+$; $--$ (cf. Theorem g, p. 146), the first sign referring to division by l_{045} and the second to division by l_{145}, and the notation so used that a point in the $++$ region is on the opposite side of l_{045} from l_{145}, and a point in the $--$ region is on the opposite side of l_{145} from l_{045}. The remaining points not on the lines are in $-+$.

Since neither l_{245} nor l_{345} can meet either l_{045} or l_{145} each must lie wholly in some one of the regions $++$, $-+$, or $--$. We have then the following cases, with the conclusion as given:

Case	l_{245} in	l_{345} in	Conclusion
1	$++$	$++$	See below.
2	$++$	$-+$	l_{045} and l_{345} between other two lines.
3	$++$	$--$	l_{045} and l_{145} " " " "
4	$-+$	$++$	l_{045} and l_{245} " " " "
5	$-+$	$-+$	l_{245} and l_{345} " " " "
6	$-+$	$--$	l_{145} and l_{245} " " " "
7	$--$	$++$	l_{145} and l_{045} " " " "
8	$--$	$-+$	l_{145} and l_{345} " " " "
9	$--$	$--$	See below.

In cases 2 to 8 inclusive, the conclusion is obvious and cases 1 and 9 differ only in notation. The proof is made only for case 9. (Fig. 7.)

Take any points P_{62} and P_{73} on l_{245} and l_{345} respectively, and P_{67} any point in $++$. If P_{62}, P_{73} and P_{67} are collinear, the theorem follows at once. If not, the segment $P_{67}P_{62}$ contains points P_{60} on l_{045} and P_{61} on l_{145}. The segment

$P_{67}P_{73}$ contains points P_{70} on l_{045} and P_{71} on l_{145}. The lines $P_{60}P_{71}$ and $P_{70}P_{61}$ determine a point P_{01}. The points P_{67} and P_{01} determine a line containing points P_{80} on l_{045} and P_{81} on l_{145} in the order $P_{67}P_{80}P_{01}P_{81}$. The segment $P_{61}P_{73}$ meets the line $P_{67}P_{01}$ in a point P_{31}. Either $P_{71}P_{31}$ and $P_{67}P_{62}$ meet in a point P_{63} or they do not. If they do meet in a point, we have the triangles $P_{60}P_{71}P_{63}$ and $P_{70}P_{61}P_{73}$ on the lines l_{045}, l_{145}, and $P_{73}P_{63}$ whose corresponding sides meet

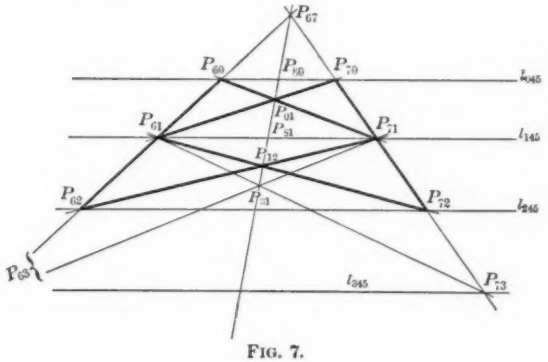

FIG. 7.

in three collinear points, P_{67}, P_{01}, P_{31}. Therefore l_{045}, l_{145}, and $P_{73}P_{63}$ are in pencil, and by Theorem IV, $P_{73}P_{63}$ is l_{345} and hence P_{63} is on l_{345}. Therefore the points P_{60}, P_{61}, P_{62}, and P_{63} are collinear, and are in one or the other of the orders $P_{60}P_{61}P_{62}P_{63}$ or $P_{60}P_{61}P_{63}P_{62}$. In either case we have the theorem. If $P_{71}P_{31}$ and $P_{67}P_{62}$ do not meet, the line $P_{71}P_{62}$ contains a point P_{12} of the line $P_{67}P_{01}$, which is in the order $P_{81}P_{12}P_{31}$; since if the points were in the order $P_{81}P_{34}P_{12}$, then $P_{71}P_{31}$ would cut the segment $P_{61}P_{62}$. Since P_{12} is on the segment $P_{81}P_{31}$, it is within the triangle $P_{71}P_{61}P_{73}$, and hence the line $P_{61}P_{12}$ must cut the segment $P_{71}P_{73}$ in a point P_{72}. The triangles $P_{62}P_{71}P_{60}$ and $P_{72}P_{61}P_{70}$ have their corresponding sides meeting in three collinear points, P_{67}, P_{01} and P_{12}. Therefore $P_{72}P_{62}$ is in pencil with l_{045} and l_{145}, and by Theorem IV, P_{72} is on l_{245}. Hence $P_{70}P_{73}$ contains P_{71} on l_{145} and P_{72} on l_{245}, and hence l_{145} and l_{245} are between l_{045} and l_{345}. This completes the proof.

COROLLARY. A triangle can always be so chosen that it will contain segments of each of the four lines l_{045}, l_{145}, l_{245} l_{345} in its interior, and such that within it no two of the lines meet, and within such a triangle the order in which the four lines are cut by a transversal is invariant.

THEOREM VI. If l_{045}, l_{145}, l_{245} are three lines in pencil, and l_{045}, l_{145}, l_{345} are three lines in pencil, then l_{045}, l_{245}, l_{345} are in pencil, and l_{145}, l_{245}, l_{345} are in pencil.

Proof. If l_{045} and l_{145} meet in a point P_k, then l_{245} and l_{345} pass through P_k, and we have at once the conclusion, by the corollary to Theorem II.

If * l_{045} and l_{145} do not meet (Fig. 8), we will consider the interior of a triangle T chosen as in the corollary to Theorem V.

All of the points chosen below are to be taken in the interior of T. Let the notation l_w, l_x, l_y, l_z be so assigned to the four lines l_{045}, l_{145}, l_{245}, l_{345} that if they are cut by any transversal in T in the points P_w, P_x, P_y, P_z these points

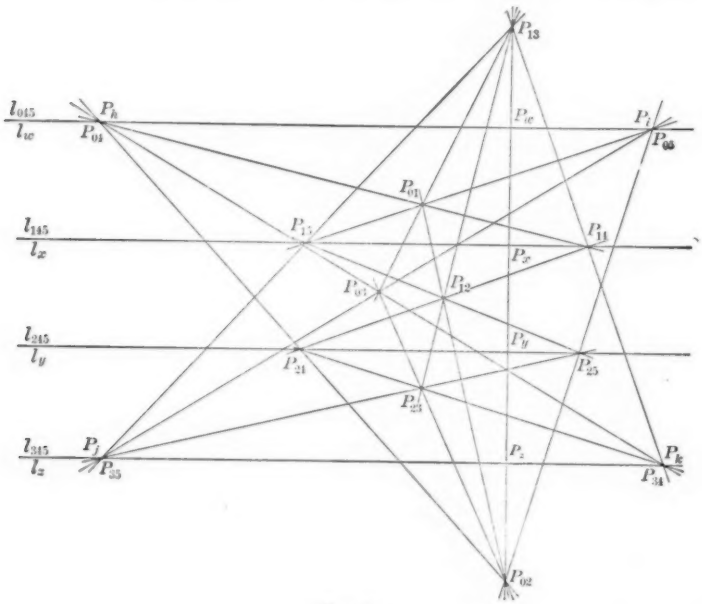

FIG. 8.

will be in the order $P_w P_x P_y P_z$. Choose P_{13} and P_{02} in the order $P_{13} P_w P_x P_y P_z P_{02}$. Then take two points P_j and P_k on l_z in the order $P_j P_z P_k$, and two points P_h and P_i on l_w, in the order $P_h P_w P_i$, and such that P_h and P_j are on the same side of the line $P_{13} P_{02}$. The line $P_{13} P_j$ meets l_{145} in a point P_{15} and l_{345} in a point P_{35}. The line $P_{13} P_k$ meets the line l_{145} in a point P_{14} and l_{345} in a point P_{34}. The line $P_{02} P_h$ meets l_{245} in a point P_{24} and l_{045} in a point P_{04}. The line $P_{02} P_i$ meets l_{245} in a point P_{25} and l_{045} in a point P_{05}. Then the points P_{14} and P_{04} are on opposite sides of the line $P_{15} P_{05}$, hence the lines $P_{14} P_{04}$ and $P_{15} P_{05}$ meet in a point P_{01}. Similarly, $P_{15} P_{25}$ and $P_{24} P_{14}$ meet in a point P_{12}; $P_{04} P_{34}$ and $P_{05} P_{35}$ meet in a point P_{03}; $P_{24} P_{34}$ and $P_{25} P_{35}$ meet in a point P_{23}.

* The configuration obtained in the proof of this theorem is a plane section of the three-space configuration containing six points, no three of them collinear. The configuration contains fifteen points and twenty lines.

Then the hypothesis l_{045}, l_{145}, l_{245} in pencil, and the hypothesis l_{045}, l_{145}, l_{345} in pencil give respectively the Desargues configurations

(A), (B)

$$\begin{array}{ccc} & Q_{45} & \\ P_{04} & P_{14} & P_{24} \\ P_{05} & P_{15} & P_{25} \\ \hline P_{12} & P_{02} & P_{01} \end{array} \qquad \begin{array}{ccc} & Q_{45} & \\ P_{04} & P_{14} & P_{34} \\ P_{05} & P_{15} & P_{35} \\ \hline P_{13} & P_{03} & P_{01} \end{array}$$

Then by using the conclusions of (A) and (B), and remembering that any three lines through a point are in pencil, we have

(C), (D)

$$\begin{array}{ccc} & Q_{14} & \\ P_{31} & P_{21} & P_{01} \\ P_{34} & P_{24} & P_{04} \\ \hline P_{02} & P_{03} & P_{23} \end{array} \qquad \begin{array}{ccc} & Q_{04} & \\ P_{30} & P_{20} & P_{10} \\ P_{34} & P_{24} & P_{14} \\ \hline P_{12} & P_{13} & P_{23} \end{array}$$

In view of (C) and (D) respectively, we have

(E), (F)

$$\begin{array}{ccc} P_{23} & P_{03} & P_{02} \\ \hline P_{04} & P_{24} & P_{34} \\ P_{05} & P_{25} & P_{35} \\ & Q_{45} & \end{array} \qquad \begin{array}{ccc} P_{23} & P_{13} & P_{12} \\ \hline P_{14} & P_{24} & P_{34} \\ P_{15} & P_{25} & P_{35} \\ & Q_{45} & \end{array}$$

The conclusion of the theorem is expressed by (E) and (F) in the configurational notation.

COROLLARY. The set of all lines through a point constitutes a pencil which is determined by any pair of the lines.

THEOREM VII. If l_{034} and l_{134} are two lines and P_{32} is a point not on either l_{034} or l_{134}, then there is one and only one line l_{234} which passes through the point P_{32} and which is in pencil with l_{034} and l_{134}.

This is a generalization of Theorem IV, in which we have the proof for all cases except when l_{034} and l_{134} do not meet, and l_{234} is between l_{034} and l_{134}. We can now prove this last case and have the more general theorem by the aid of Theorem VI. We will first prove that there can not be two such lines. Suppose l_{234} and l_{534} are two distinct lines through P_{34} each in pencil with l_{034} and l_{134}. Then by Theorem VI, l_{034}, l_{234}, and l_{534} are in pencil, and by Theorem I, l_{034}, l_{134}, l_{534} must meet in P_{32}, but this contradicts the hypothesis that l_{034} and l_{134} do not meet, and hence there can not be two such lines.

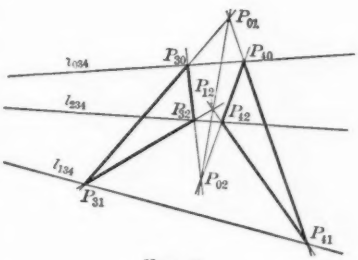

FIG. 9.

One line, and by this proof the only line l_{234}, through P_{32} and in pencil with l_{034} and l_{134} will now be exhibited. (Fig. 9.) Take any point P_{30} on l_{034}, and any point P_{02} not on l_{134}, in the order $P_{30}P_{32}P_{02}$, and such that the segment $P_{32}P_{02}$ contains no point of l_{134}. Then take P_{31} on the line l_{134}, but not on the line $P_{30}P_{32}$, and then any point P_{01} in the order $P_{31}P_{30}P_{01}$. Choose $P_{40} \neq P_{30}$ on l_{034} but not on the line $P_{01}P_{02}$, and any point $P_{41} \neq P_{31}$ on the line l_{134}, and such that these points are in the order $P_{01}P_{40}P_{41}$. Then $P_{31}P_{32}$ and $P_{01}P_{02}$ meet in a point P_{12}, and the segment $P_{40}P_{02}$ and the segment $P_{41}P_{12}$ meet in a point P_{42}. Hence we have the Desargues configuration:

$$\begin{array}{ccc} P_{12} & P_{02} & P_{01} \\ \hline P_{30} & P_{31} & P_{32} \\ P_{40} & P_{41} & P_{42} \\ & Q_{34} & \end{array}$$

whence the line $P_{32}P_{42}$ is the line l_{234} of the theorem.

COROLLARY. Any two lines determine a pencil.

In view of Theorem VII we can now speak of P_xQ_y, where P_x is not the center of Q_y as a definite line, viz., the line of the pencil Q_y which passes through P_x. We will also speak of Q_x as collinear with P_y and P_z if the line P_xP_y belongs to the pencil Q_z; also if the line P_xQ_y is a line of the pencil Q_z, $P_xQ_yQ_z$ will be said to be collinear. Collinearity, when used in this sense, of course, does *not imply order relations*. A line can belong to any number of pencils, viz., the pencils determined by it and each of the lines through some point not on it. But two pencils can not have more than one *line* in common, by Theorem VII. Since a point Q_x always determines uniquely a pencil, viz., the pencil of which it is the center, we may use any point as a pencil, when we mean this pencil. Points will be so used, in speaking of ranges, as in the definition of "in range," points might have been used for the pencils, as indicating the pencils of which they are centers.

SECTION II.

Consequences of Axioms I–XI.

The theorems of Section 2 will have to do with ranges and their properties. It is evident from the definitions of pencils and "in range," that any three points of a line are the centers of three pencils which are in range. The two triangles whose sides are lines of three pencils in range, and the joins of whose vertices are in pencil, will be said to be *perspective triangles*.

Suppose Q_1, Q_2, Q_3 are any three distinct pencils, and T is any triangle not containing in its interior or on any of its sides any segment common to any two of these pencils. Let $P_aP''_bP_c$ be any three points in T such that $P_aP''_b$ is a

line of Q_1 and P_aP_c is a line of Q_2. (Fig. 10.) Then the lines P_cQ_1 and $P_b''Q_2$ meet in a point P_d of T, or do not. If not, let any point of P_cQ_1 on the same side of P_aP_c that P_b is, be called P_d; then P_dQ_2 will meet P_aP_b'' in a point P_b which may be used for P_b''.

Such a figure ($P_aP_bP_cP_d$) will be called a *quadrilateral* in T on Q_1 and Q_2.

Some one at least of the lines P_aQ_3, P_bQ_3, P_cQ_3, P_dQ_3 enters the quadrilateral $P_aP_bP_cP_d$, and by a proper assignment of notation it may be considered P_aQ_3. Then P_aQ_3 will meet either the segment P_cP_d or the segment P_bP_d in a point P_d'' or P_d' respectively. In the first case, the line $P_d''Q_2$ meets the segment P_aP_b in a point P_b', and P_d'' and P_b' may be used as a new P_d and P_b, respectively. In the second case, the line $P_d'Q_1$ meets the segment P_aP_c in a point P_c', and P_d' and P_c' may be used as P_d and P_c respectively, and hence the points $P_aP_bP_cP_d$ can be so chosen that they form a quadrilateral on Q_1 and Q_2, and so that P_aP_d is a line of Q_3. Such a figure will be called a quadrilateral on Q_1, Q_2 and Q_3, in T. Any line l_1 of Q_1 and any line l_2 of Q_2, which are within the quadrilateral, meet each other in a point of the quadrilateral.

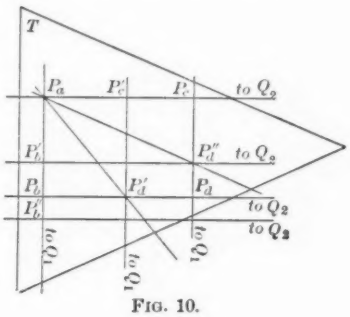

FIG. 10.

The notation for a Desargues configuration will sometimes be written, when the notation is in the form of Definition 7, D. C. (34)–(012), and when in the form of Axiom XI, D. C. (012)–(34), for shortness in writing as this may readily be expanded to the previous form.

THEOREM VIII. If Q_{01} and Q_{02} are any two distinct pencils, and l is any line not belonging to either Q_{01} or Q_{02}, then l is in one and only one pencil Q_{12} which is in range with Q_{01} and Q_{02}. (Fig. 11.)

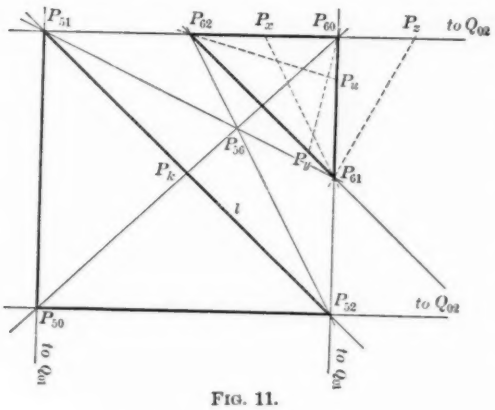

FIG. 11.

Proof. Let T be any triangle containing in its interior a segment of l, but no segment of a line common to Q_{01} and Q_{02}. Choose P_{51} and P_{52} on l, in T, and such that $P_{51}Q_{01}$ and $P_{52}Q_{02}$ meet in a point P_{50} of T, and such that $P_{50}Q_{01}$ and $P_{51}Q_{02}$ meet in a

point P_{60} of T. Then the segments $P_{51}P_{52}$ and $P_{60}P_{50}$ meet in a point P_k; choose P_{56} in the order $P_kP_{56}P_{60}$. Then $P_{52}P_{56}$ meets the segment $P_{51}P_{60}$ in a point P_{62}, and $P_{51}P_{56}$ meets the segment $P_{52}P_{60}$ in a point P_{61}. Then from the triangles $P_{50}P_{51}P_{52}$ and $P_{60}P_{61}P_{62}$ perspective in P_{56}, $P_{51}P_{52}$ and $P_{61}P_{62}$ determine a pencil Q_{12} which is in range with Q_{01} and Q_{02}.

Suppose there is a pencil Q'_{12}, distinct from Q_{12}, such that l is a line of Q'_{12}, and such that Q_{01}, Q_{02}, and Q'_{12} are in range. Then at least one of the lines $P_{60}Q'_{12}$, $P_{61}Q'_{12}$, $P_{62}Q'_{12}$ must enter the triangle $P_{60}P_{61}P_{62}$, unless $P_{60}Q'_{12}$ is identical with $P_{60}Q_{01}$ or $P_{60}Q_{02}$. But either of these was avoidable by a new choice of P_{52} and P_{51}, since P_{52} and P_{51} might have been any points on the segment $P_{51}P_{52}$.

Case I. Suppose $P_{60}Q'_{12}$ cuts the segment $P_{61}P_{62}$ in P_y. Then choose a point P_z in the order $P_{51}P_{60}P_z$ such that $P_zQ'_{12}$ passes through P_{61}, or a point P'_{61} on the segment $P_{61}P_{60}$. Then since P_{56} was chosen arbitrarily on the segment P_kP_{60}, it may be chosen so that for some point P_z in order $P_{51}P_{60}P_z$, $P_zQ'_{12}$ passes through P_{61}. Then since $Q_{01}Q_{02}Q'_{12}$ are in range, the triangles $P_{51}P_{52}P_{50}$ and $P_{61}P_zP_{60}$ are in perspective. But $P_{61}P_{51}$ and $P_{60}P_{50}$ meet in P_{56}, and $P_{52}P_z$ does not pass through P_{56}. Hence this case is impossible.

Case II. $P_{61}Q'_{12}$ meets the segment $P_{60}P_{62}$ in a point P_x. Then since Q_{01}, Q_{02}, and Q'_{12} are in range, by hypothesis, $P_{60}P_{61}P_x$ and $P_{50}P_{51}P_{52}$ are in perspective. But $P_{60}P_{50}$ and $P_{61}P_{51}$ meet in P_{56} and P_xP_{52} does not pass through P_{56}. Hence this case is impossible.

Case III. $P_{62}Q'_{12}$ meets the segment $P_{61}P_{62}$ in P_u. The proof in this case is identical with that of Case II, except for notation. Hence there can be no such pencil Q'_{12}, and the theorem is proved.

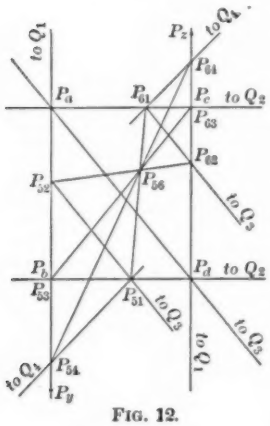

FIG. 12.

THEOREM IX. If Q_1, Q_2, Q_3 and Q_4 are four distinct pencils, and Q_1, Q_2, Q_3 are in range, and Q_1, Q_2, Q_4 are in range, then also Q_1, Q_3, Q_4 are in range. (Fig. 12.)

Proof. Let $P_aP_bP_cP_d$ be a quadrilateral on Q_1, Q_2, Q_3 in a region T not containing any point of a line common to any pair of the pencils Q_1, Q_2, Q_3, Q_4. Let P_y and P_z be any points in T in the orders $P_aP_bP_y$ and $P_dP_cP_z$ respectively. Let P_{51} be any point on the segment P_bP_d such that $P_{51}Q_4$ meets the segment P_aP_y in a point P_{54}. Let P_{61} be any point on the segment P_aP_c, such that $P_{61}Q_4$ meets the segment P_dP_z in a point P_{64}. $P_{51}Q_3$ meets the segment P_aP_b in a point P_{52}, and $P_{61}Q_3$ meets the segment P_cP_d in a point P_{62}. Since P_{61}, P_{51} are on one pair of opposite sides of the quadrilateral, and

P_{62}, P_{52} are on the other pair of opposite sides, $P_{61}P_{51}$ and $P_{62}P_{52}$ meet in a point P_{56}, of the interior of the quadrilateral. If P_b and P_c be called P_{53} and P_{63} respectively, we have the Desargues configuration (123)–(56), whence $P_{53}P_{63}$ also passes through the point P_{56}. But from the hypothesis that Q_1, Q_2 and Q_4 are in range, and the triangles $P_{51}P_{52}P_{54}$ and $P_{61}P_{62}P_{64}$, we have the Desargues configuration (124)–(56), whence $P_{54}P_{64}$ also passes through P_{56}. Hence, from triangles $P_{51}P_{53}P_{54}$ and $P_{61}P_{63}P_{64}$, perspective in P_{56}, we have the Desargues configuration (56)–(134), whence Q_1, Q_3, and Q_4 are in range, and the theorem is proved.

COROLLARY. Any two pencils of a range determine the range.

THEOREM X. If three points P_{01}, P_{02}, P_{12} are in range, they are collinear.

Proof. (See Fig. 13.) Let P_{30} be a point not on any of the lines $P_{01}P_{02}$, $P_{01}P_{12}$, $P_{02}P_{12}$. Take P_{31} on the line $P_{30}P_{01}$ such that the line $P_{31}P_{12}$ will

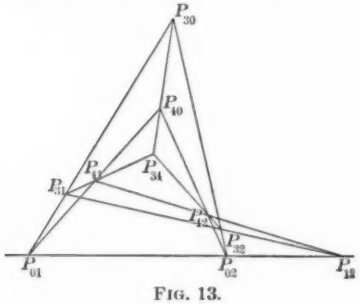

FIG. 13.

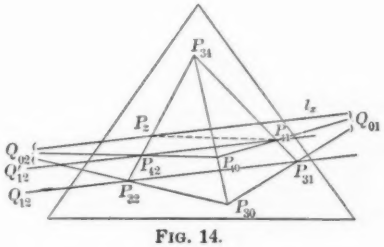

FIG. 14.

meet the segment $P_{30}P_{02}$ in a point P_{32}, and such that no point of any of the lines $P_{01}P_{02}$, $P_{01}P_{12}$, $P_{02}P_{12}$ is within the triangle $P_{30}P_{31}P_{32}$. Let P_{34} be any point within the triangle $P_{30}P_{31}P_{32}$. Take P_{40} in the order $P_{30}P_{40}P_{34}$. Then the line $P_{40}P_{01}$ will meet the segment $P_{31}P_{34}$ in a point P_{41}, and the line $P_{40}P_{02}$ will meet the segment $P_{34}P_{32}$ in a point P_{42}. But the lines $P_{42}P_{41}$ and $P_{32}P_{31}$ determine a pencil Q'_{12}, which, by definition, is in range with P_{01} and P_{02}. Hence by Theorem VIII Q_{12} and Q'_{12} are identical, and we have the theorem.

THEOREM XI. If Q_{01}, Q_{02}, Q_{12} are in range and l_x is a line belonging to Q_{01} and Q_{02}, then l_x belongs also to Q_{12}. (Fig. 14.)

Proof. Let T be a triangle containing in its interior a segment of the line l_x, but no center of any one of the three pencils. Let P_{30} be any point in T not on l_x, and P_{31} a point in T, on line $P_{30}Q_{01}$, and P_{32} a point in T on $Q_{02}P_{30}$, such that $P_{31}P_{32}$ is a line of Q_{12}, and P_{30}, P_{31}, P_{32} are all on the same side of l_x. Take P_{34} on opposite side of l_x from P_{30}, not on $P_{32}Q_{12}$. Suppose now l_x is *not* a line of Q_{12}; let P_z be the point in which $P_{32}P_{34}$ meets l_x. Then P_zQ_{12} will meet the segment $P_{34}P_{31}$ in a point P_{41}, not on l_x. $P_{41}Q_{01}$ meets $P_{30}P_{34}$ in a point P_{40}, not on l_x. $P_{40}Q_{02}$ meets $P_{32}P_{34}$ in a point P_{42}, not on l_x. Hence

$P_{31}P_{32}$ and $P_{41}P_{42}$ determine a pencil Q'_{12} which is in range with Q_{01} and Q_{02}. By Theorem VIII Q'_{12} is Q_{12}. But $P_{41}P_x$ was supposed to be a line of Q_{12}, and that would make P_{41} the center of Q_{12}. But Q_{12} does not have its center in T, and P_{41} is in T, hence l_x must be a line of Q_{12}.

THEOREM XII. If l_x is a line of a pencil Q_{12}, and P_{02} and P_{01} are any two distinct points of l_x not the center of Q_{12}, then Q_{12}, P_{02} and P_{01} are in range.

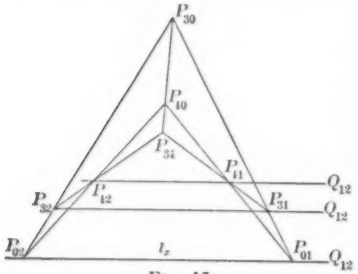

FIG. 15.

Proof. (See Fig. 15.) Let P_{30} be any point not on l_x, and P_{31} and P_{32} two points on the same side of l_x that P_{30} is, and such that $P_{30}P_{31}$ is a line of Q_{01}, $P_{30}P_{32}$ is a line of Q_{02}, and $P_{31}P_{32}$ is a line of Q_{12}. Let P_{34} be any point within the triangle $P_{30}P_{31}P_{32}$. Let P_{40} be any point in the order $P_{30}P_{40}P_{34}$. Then $P_{40}Q_1$ meets the segment $P_{31}P_{34}$ in a point P_{41}, and $P_{40}Q_{02}$ meets the segment $P_{34}P_{32}$ in a point P_{42}. Then the line $P_{42}P_{41}$ belongs to the pencil Q_{12}, from the Desargues configuration, (043)–(12), and hence the theorem is true.

THEOREM XIII. If l_x belongs to Q_{01} and Q_{02}, then any other pencil Q_{12} to which l_x belongs is in range with Q_{01} and Q_{02}. Let P_x and P_y be two points of l_x not centers of Q_{01} or Q_{02} or Q_{12}. Then $P_xP_yQ_{01}$, $P_xP_yQ_{02}$, $P_xP_yQ_{12}$ are in range, by Theorem XII. Then by Theorem IX, $P_xQ_{01}Q_{02}$ and $P_xQ_{01}Q_{12}$ and therefore Q_{01}, Q_{02}, Q_{12} are in range.

THEOREM XIV. Any two distinct ranges r_{023} and r_{123} have in common one and only one pencil, Q_{23}. (Fig. 16.)

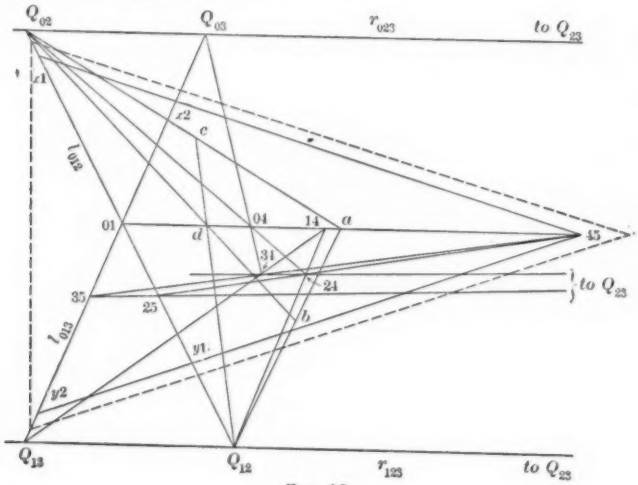

FIG. 16.

Proof. Through any point P_{01}, not a point of either r_{023} or r_{123}, draw lines l_{013} and l_{012}, which with r_{023} and r_{123} determine respectively the pencils Q_{03} and Q_{13}, Q_{02} and Q_{12}. Choose P_{45}, P_{x2} and P_{y1} such that P_{x2} is on the line l_{013} and P_{y1} is on the line l_{012}, and such that $P_{45}P_{x2}$ meets l_{012} in a point P_{x1} and $P_{45}P_{y1}$ meets l_{013} in a point P_{y2} in the orders $P_{x1}P_{01}P_{y1}$ and $P_{x2}P_{01}P_{y2}$, and such that all of these points lie within a triangle containing no points of r_{123} or r_{023}. Take a quadrilateral $P_aP_bP_cP_d$ on Q_{12} or Q_{02}, and P_{01}, and such that P_a and P_d are on the segment $P_{01}P_{45}$, and such that P_cP_{45} and P_bP_{45} are lines within the triangles $P_{45}P_{x1}P_{01}$ and $P_{45}P_{y2}P_{01}$ respectively. Take P_{04} and P_{14} on the segment P_aP_d, and such that $P_{04}P_{03}$ and $P_{14}Q_{13}$ meet at a point P_{34} within the quadrilateral $P_aP_bP_cP_d$; $P_{04}Q_{02}$ and $P_{14}Q_{12}$ meet at a point P_{24} within the quadrilateral. Then $P_{45}P_{24}$ meets l_{012} in a point P_{25} and $P_{45}P_{34}$ meets the line l_{013} in a point P_{35}. Then $P_{25}P_{35}$ and $P_{24}P_{34}$ determine a pencil Q_{23} which is in range with $Q_{02}Q_{03}$ from D. C. (45)–(023), and in range with $Q_{12}Q_{13}$ from D. C. (45)–(123).*

That there is only one such pencil is an immediate consequence of Theorem IX. For if there were two such pencils, then these would determine any range containing them, and hence the ranges r_{023} and r_{123} would not be distinct.

THEOREM XV. If any ten pencils Q_{ij} $(i, j = 1, \cdots, 5;\ i \neq j)$ are such that no four of them are in range, and $Q_{ij}\,Q_{ik}\,Q_{jk}$ $(i,\ j,\ k = 1,\ 2,\ 3,\ 4,\ 5;\ i \neq j \neq k$, and $i, j, k \neq 1, 2, 3$ in any order) are in range r_{ijk}, then also Q_{12}, Q_{13} and Q_{23} are in range r_{123}, and conversely.

The theorem is the general Desargues theorem, pencils and ranges replacing points and lines, and forming the configuration $\left|\begin{smallmatrix}10 & 3\\ 3 & 10\end{smallmatrix}\right|$. This configuration, for the purpose of the proof, is enlarged to a configuration $\left|\begin{smallmatrix}21 & 3\\ 5 & 35\end{smallmatrix}\right|$, P_{ij} or Q_{ij}, l_{ijk} or r_{ijk} $(i,\ j,\ k = 1, \cdots, 7)$, the new lines or points being within or crossing a region T, say the inside of a triangle, containing no point of the original ranges. The eleven points added form two perspective five-points $P_{1,2,3,4,5-6}$ and $P_{1,2,3,4,5-7}$, perspective from a point P_{67}.

Choose P_{36}, any point in T (Fig. 17), and draw $P_{36}\,Q_{3j}$ $(j = 1, 2, 4)$; in T, choose points P_{16} and P_{26}, on $P_{36}Q_{13}$ and $P_{36}Q_{23}$ respectively, and such that $P_{16}Q_{14}$ and $P_{26}Q_{24}$ meet in a point P_{46} of T, and $P_{26}Q_{25}$ and $P_{16}Q_{15}$ meet in a point P_{56} of T. Take P_{67} in T, but neither collinear with any two points, nor on any line yet chosen, and draw $P_{67}P_{6j}$ $(j = 1 \cdots 5)$. Take P_{17} on the segment $P_{67}P_{16}$, and draw $P_{17}Q_{12}$. This will cut the segment $P_{67}P_{26}$ in a point P_{27}. $P_{17}P_{14}$ meets segment $P_{67}P_{46}$ in a point P_{47}, which is also on the line $P_{27}Q_{24}$, from the D. C. (67)–(124). [That is, $P_{27}P_{47}$ and $P_{26}P_{46}$ determine a pencil which is in range with Q_{14} and Q_{12}, and since Q_{24} is a pencil of the line $P_{26}P_{46}$, it is this pencil.] Also $P_{17}Q_{15}$ meets the segment $P_{56}P_{67}$ in a point P_{57}, and from

*The configuration used in the proof of this theorem is a degenerate case of the configuration $\left|\begin{smallmatrix}15 & 4\\ 3 & 20\end{smallmatrix}\right|$, of notation P_{ij}, l_{ijk} $(i, j, k = 0, \cdots, 5)$ in which the points P_{01}, P_{03} and P_{15} are coincident.

the D. C. (67)–(125), $P_{26}P_{56}$ and $P_{27}P_{57}$ determine a pencil which is in range with Q_{15} and Q_{12}, and is therefore Q_{25}. From the D. C. (67)–(ijk), $(i=1, 2;$ $j, k=3, 4, 5, j \neq k)$ $P_{j6}P_{k6}$ and $P_{j7}P_{k7}$ belong to Q_{jk}. Hence Q_{34}, Q_{35},

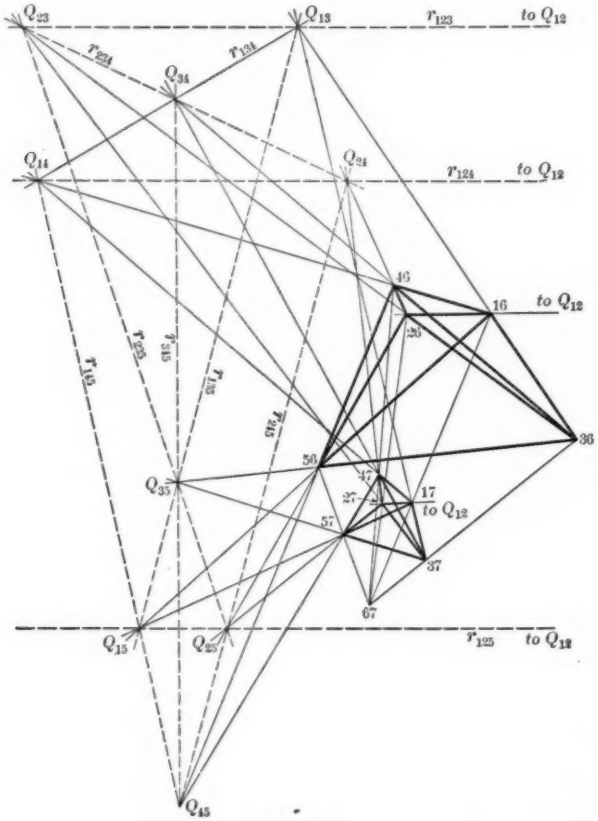

FIG. 17.

Q_{45} are in range, from the D. C. (67)–(345), and we have established the first part of the theorem.

The same configuration serves for the proof of the converse theorem, if the points are chosen in a slightly different order. Choose in T a point P_{36}, then P_{46} and P_{56} such that $P_{46}P_{56}$ is a line of Q_{56}, $P_{56}Q_{25}$ and $P_{46}Q_{24}$ meet in a point P_{26} of T, and $P_{56}Q_{15}$ and $P_{46}Q_{14}$ meet in a point P_{16} of T. Choose P_{67} in T, not on any line or range of the figure so far determined. On the segment $P_{67}P_{36}$ choose P_{37}. Then $P_{37}Q_{35}$ meets the segment $P_{56}P_{67}$ in a point P_{57}. The line $P_{57}Q_{45}$ meets the segment $P_{46}P_{67}$ in a point P_{47}. Then from

the D. C. (67)–(345), $P_{37}P_{47}$ is a line of Q_{34}. Then $Q_{14}P_{47}$ meets the segment $P_{16}P_{67}$ in a point P_{17}. From D. C. (67)–(145) and Theorem VIII, $P_{17}P_{57}Q_{15}$ are collinear. We have now hypotheses for the configuration (67)–(134) and (67)–(135), from which, and Theorem VIII, we have the collinearities $P_{16}P_{36}Q_{13}$, $P_{17}P_{37}Q_{13}$. The line $P_{47}Q_{24}$ meets the segment $P_{26}P_{67}$ in a point P_{27}, and from the D. C. (67)–(245) we have $P_{27}P_{57}Q_{25}$ collinear. From the D. C.'s (67)–(234) and (67)–(235) we have the collinearities $P_{26}P_{36}Q_{23}$ and $P_{27}P_{37}Q_{23}$.

But from the D. C.'s (67)–(123), (67)–(124), (67)–(125), the lines $P_{16}P_{26}$ and $P_{17}P_{27}$ determine a pencil Q_{12} which is in range with $Q_{13}Q_{23}$, $Q_{14}Q_{24}$, $Q_{15}Q_{25}$ respectively, and this proves the theorem.

Section III.

Generalization of Order.

In this section it is shown that if some one range of pencils is omitted from the set of all pencils, the remaining subset of pencils has the property, that, with a suitable definition of order, its elements satisfy the Axioms I–XI, a well-known property of the elements of a projective plane.

Before giving the definition of order it will be useful to prove certain lemmas.

Lemma I. If Q_h, Q_i, Q_j, Q_k are any four distinct pencils of a range r_{hijk}, and $T_x \equiv P_{xhi}P_{xhj}P_{xij}$ is a triangle whose sides belong to Q_h, Q_i, Q_j respectively (i. e., the line $P_{xhi}P_{xhj}$ belongs to the pencil Q_h, etc.), all of whose vertices are on the same side of r_{hijk}, then of the three statements

1) $P_{xhi}Q_k$ cuts the segment $P_{xjh}P_{xij}$ in a point P_{xjk};
2) $P_{xhj}Q_k$ cuts the segment $P_{xhi}P_{xij}$ in a point P_{xik};
3) $P_{xij}Q_k$ cuts the segment $P_{xhi}P_{xhj}$ in a point P_{xhk};

one and only one is true.

Proof. Since the vertices of the triangle are all on one side of the range r_{hijk}, no side of the triangle is r_{hijk}. Then the lines $P_{xhi}Q_k$, $P_{xhj}Q_k$, $P_{xij}Q_k$ are all distinct lines of a pencil, whose center, if it exists, is not within the triangle. If the pencil has a center, the lemma follows at once from theorem g of the introduction. If it has no center, it follows from Theorem III. It is evident that by a proper assignment of the notation Q_h, Q_i, Q_j the true statement may be made any desired one of the three.

Lemma II. If T_3 and T_4 are two triangles satisfying the hypothesis of Lemma I, and such that in the triangle T_3, the line $P_{3hj}Q_k$ meets the segment $P_{3hi}P_{3ij}$ in a point P_{3ik}, then also the line $P_{4hj}Q_k$ meets the segment $P_{4hi}P_{4ij}$ in a point P_{4ik}.

Proof. Case I. (Fig. 18.) When no pair of corresponding sides of the triangles T_3 and T_4 lie in the same line, and no vertices are in common. The lines

$P_{3hi}P_{4hi}$, $P_{3hj}P_{4hj}$, $P_{3ij}P_{4ij}$ are in pencil Q_{34}. The line $P_{3ik}Q_{34}$ meets the segment $P_{4hi}P_{4ij}$ in a point P_{4ik}, by theorems on division of a plane by lines and Theorem I, if Q_{34} has a center, and by Theorem IV, if Q_{34} has no center. But

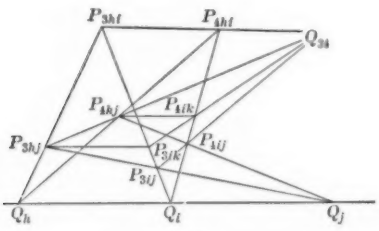

FIG. 18.

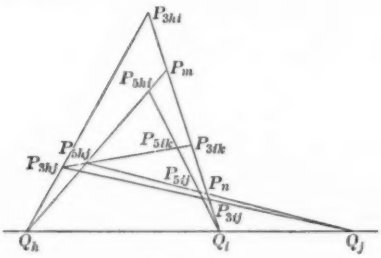

FIG. 19.

$P_{4hj}P_{4ik}$ and $P_{3hj}P_{3ik}$ determine a pencil in range with Q_h and Q_i, and hence Q_k.

Case II. (Fig. 19.) In case some parts of the triangles T_3 and T_4 are coincident, we can make the proof as follows. It is always possible to obtain a triangle T_5, satisfying hypothesis of Lemma II, all of whose parts are distinct from all the parts of T_3 and T_4, and such that $P_{5hj}Q_k$ meets the segment $P_{5hi}P_{5ij}$ in a point P_{5ik}, whence the theorem follows by Case I. Such a triangle T_5 is obtained as follows. Within the triangle T_3 and on the segment $P_{3hj}P_{3ik}$ but on no side of T_4, choose a point P_{5hj}. The line $P_{5hj}Q_h$ will meet the segment $P_{3hi}P_{3ik}$ in a point P_m, and the line $P_{5hj}Q_j$ will meet the segment $P_{3ik}P_{3ij}$ in a point P_n. Choose a point P_{5hi} on the segment $P_{5hj}P_m$ and not on any line of T_4. Then $P_{5hi}Q_i$ will meet the lines $P_{5hj}P_n$ and $P_{5hj}P_{3ik}$ in points P_{5ij} and P_{5ik} respectively, and the triangle $P_{5hj}P_{5hi}P_{5ij}$ is the triangle required.

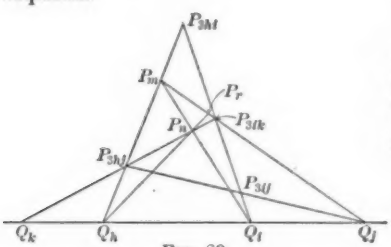

FIG. 20.

LEMMA III. (Fig. 20.) If Q_h, Q_i, Q_j, Q_k are four distinct pencils of a range, r_{hijk}, and T_3 is a triangle satisfying the hypotheses of Lemma II, then in any triangle

$$T_x \equiv \left\{ \begin{matrix} P_{xhi}P_{xhk}P_{xik} \\ P_{xhj}P_{xhk}P_{xjk} \\ P_{xij}P_{xik}P_{xjk} \end{matrix} \right\}$$

all of whose vertices are on the same side of r_{hijk}, and whose sides belong to the pencils

$$\left\{ \begin{matrix} Q_hQ_iQ_k \\ Q_hQ_jQ_k \\ Q_iQ_jQ_k \end{matrix} \right\},$$

respectively, the line and segment

$$\left\{\begin{matrix} P_{xik}\,Q_j \\ P_{xhj}\,Q_i \\ P_{xik}\,Q_h \end{matrix}\right\}, \qquad \left\{\begin{matrix} P_{xhi}\,P_{xhk} \\ P_{xhk}\,P_{xjk} \\ P_{xij}\,P_{xjk} \end{matrix}\right\},$$

meet in a point

$$\left\{\begin{matrix} P_{xhj} \\ P_{xki} \\ P_{xjh} \end{matrix}\right\}.$$

Proof. The line $Q_j P_{3ik}$ can not meet the segment $P_{3hj} P_{3ij}$, hence meets the segment $P_{3hj} P_{3hi}$ in a point P_m, and hence from the triangle $P_{3hi} P_{3hj} P_{3ik}$ and Lemma II we have the first part of the lemma. Similarly, $P_m Q_i$ must meet the segment $P_{3hj} P_{3ik}$ in a point P_n, and hence from the triangle $P_m P_{3hj} P_{3ik}$ and Lemma II we have the second part of the lemma. Also $P_n Q_h$ must meet the segment $P_m P_{3ik}$ in a point P_r, and from the triangle $P_m P_n P_{3ik}$ and Lemma II we have the third part of the lemma.

The range of pencils omitted from the set Q_k will be called r_∞, and pencils not on r_∞ will be denoted by the notation S_k.

Definition of Order. (Fig. 21.) Three pencils have order relations if and only if they are distinct, lie in a range, and no one of them lies on r_∞. Three such pencils S_i, S_j, S_k, of a range r_{ijk} meeting r_∞ in Q_∞, are in the order $S_i S_j S_k$ if and only if in a triangle $P_{xij} P_{xik} P_{xjk}$, whose sides belong to the pencils S_i, S_j, S_k respectively, and all of whose vertices are on the same side of r_∞ and r_{ijk}, the line $P_{xik} Q_\infty$ meets the segment $P_{xij} P_{xjk}$ in a point $P_{j\infty}$.

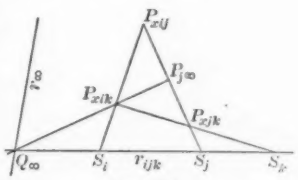

Fig. 21.

It will now be shown that the set of pencils S, subject to the above definition of order, satisfies the Axioms I–XI.

Axiom I is evident, since every point is the center of a pencil.

Axiom II is evident from the symmetry of the definition of order.

Axiom III follows from Lemma I.

Axiom IV is evident from definition.

Axiom V is evident from definition and Lemma III.

Axiom VI is a consequence of Theorem IX.

Axiom VII is obvious.

Axioms IX, X, XI are evident, since all points are centers of pencils, and they hold for the set of *all* pencils.

Axiom VIII. (Fig. 22.) This will be paraphrased as follows. If in the set S_k, S_{145}, S_{245}, and S_{345} are any three pencils not in range, and S_{235} is a pencil in the order $S_{245} S_{345} S_{235}$, and S_{135} is a pencil in the order $S_{145} S_{135} S_{345}$,

then there exists a pencil S_{125} which is in the order $S_{145}\,S_{125}\,S_{245}$, and which is in range with S_{135} and S_{235}.

Proof. The ranges $S_{135}\,S_{235}$ and $S_{145}\,S_{245}$ are distinct, and hence they determine a pencil Q_{125}. Select a triangle T containing in its interior no point of any of the ranges $S_{245}\,S_{125}$, $S_{245}\,S_{345}$, $S_{145}\,S_{345}$, r_∞. All points selected below are to be taken within the triangle T. Call the pencils common to r_∞ and $S_{145}\,S_{245}$,

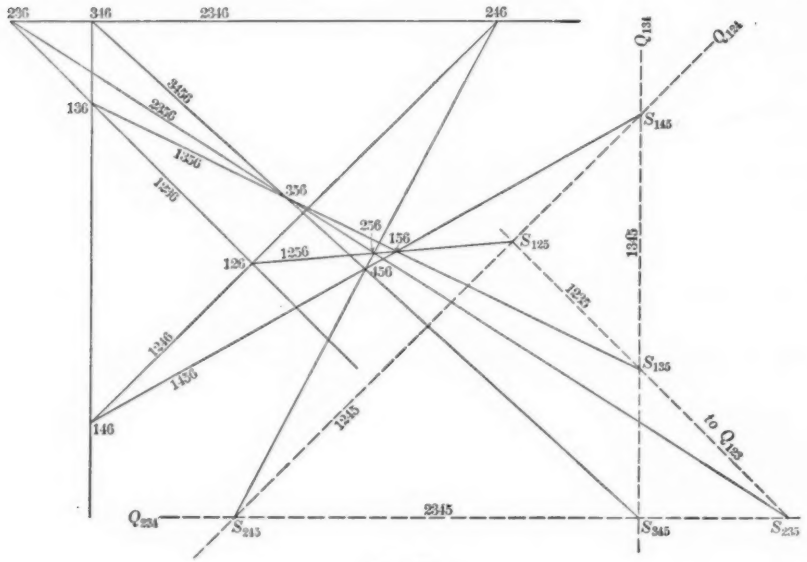

FIG. 22.

$S_{245}\,S_{345}$, $S_{145}\,S_{345}$, $S_{135}\,S_{235}$, respectively Q_{124}, Q_{234}, Q_{134}, Q_{123}. Choose in T a point P_{456}, and then points P_{146}, P_{246}, P_{346} on the lines $P_{456}\,S_{145}$, $P_{456}\,S_{245}$, $P_{456}\,S_{345}$, respectively, and such that the lines $P_{346}\,P_{246}$, $P_{346}\,P_{146}$, and $P_{146}\,P_{246}$ belong to the pencils Q_{234}, Q_{134}, and Q_{124}, respectively. From the order $S_{245}\,S_{345}\,S_{235}$ of the hypothesis, $P_{346}\,S_{235}$ meets the segment $P_{456}\,P_{246}$ of the triangle $P_{246}\,P_{346}\,P_{546}$. Hence it is possible to find in T points $P_{236}\,P_{356}\,P_{256}$ in the orders $P_{236}\,P_{346}\,P_{246}$, $P_{346}\,P_{356}\,P_{456}$, $P_{246}\,P_{256}\,P_{456}$, $P_{236}\,P_{356}\,P_{256}$. (Equally well might another set of orders have been secured.)

Then $P_{356}\,S_{135}$ meets $P_{456}\,S_{145}$ in a point P_{156} of T or does not. If it does, P_{156} is in the order $P_{146}\,P_{456}\,P_{156}$, from the order $S_{145}\,S_{135}\,S_{345}$ of the hypothesis and the triangle $P_{346}\,P_{146}\,P_{456}$. If not, choose in T a point P'_{156} in the order $P_{146}\,P_{456}\,P'_{156}$, and from the order $S_{145}\,S_{135}\,S_{345}$ of the hypothesis, and the triangle $P_{346}\,P_{146}\,P_{456}$, by Lemmas I, III, $P_{456}\,S_{135}$ meets the segment $P_{346}\,P_{146}$, and $P'_{156}\,S_{135}$ will meet the segment $P_{456}\,P_{356}$ in a point P'_{356}. The line

$P'_{356}S_{235}$ will meet the segment $P_{456}P_{256}$ in P'_{256}, and will or will not meet $P_{346}P_{246}$ in a point P_x of T. Take P'_{236} in the order $P'_{256}P'_{356}P'_{236}$, and in the order $P'_{356}P'_{236}P_x$ if P_x exists, and such that the line $P'_{236}Q_{234}$ meets the segment $P_{346}P_{146}$ in a point P'_{346}, and the segment $P_{146}P_{246}$ in P'_{246}. In this case use the points marked with primes in the further discussion, where points without the primes are used. The line $P_{356}S_{135}$ will meet the segment $P_{346}P_{146}$ in a point P_{136}. The sets of three pencils, $P_{256}P_{356}P_{456}$ and $S_{215}S_{315}S_{415}$ have corresponding ranges belonging to the pencils Q_{345}, Q_{245}, and Q_{235} of the range r_{2345}, and the lines $S_{456}S_{145}$ and $P_{356}S_{135}$ meet in P_{156}. Hence the line $P_{256}S_{125}$ also passes through the point P_{156} (by Theorem XIII), and from the order $P_{456}P_{356}P_{346}$, the point P_{136} is in the orders $P_{156}P_{356}P_{136}$ and $P_{346}P_{136}P_{146}$. Also, the line $P_{236}P_{136}$ meets the line $P_{146}P_{246}$ in a point P_{126} in the orders $P_{236}P_{136}P_{126}$ and $P_{146}P_{126}P_{246}$. From the sets of three pencils $P_{136}P_{236}P_{436}$ and $S_{135}S_{235}S_{435}$ "perspective" in P_{356}, the line $P_{136}P_{236}$ belongs to the pencil Q_{123}. From the sets of three pencils $S_{125}S_{325}S_{425}$ and $P_{126}P_{326}P_{426}$, whose sides belong in pairs to the three pencils Q_{124}, Q_{234}, Q_{123}, respectively, P_{126} is on the line $P_{256}P_{156}$. From the triangle $P_{156}P_{256}P_{456}$, the line $P_{456}Q_{124}$ cuts the segment $P_{156}P_{256}$, since it is not either side line of the triangle, and since it can not meet the segment $P_{146}P_{246}$. Hence if Q_{125} is not on r_∞, it is S_{125} and in the order $S_{145}S_{125}S_{245}$. That it is not on r_∞ may be seen as follows: $S_{245}S_{145}$ meets r_∞ in Q_{124}, of which $P_{146}P_{246}$ is a line. If this line were the same as $P_{126}P_{256}$, P_{156} would be on the line $P_{146}P_{246}$, contrary to the hypothesis. Hence the axiom holds.

The configuration used in the proof may be indicated schematically by $\left|\begin{smallmatrix}20 & 4\\ 3 & 15\end{smallmatrix}\right|$, as may readily be verified from the figure.

Section IV.

In terms of the plane set of pencils S, already exhibited in Section III, a new set of elements will now be defined which will be shown to form a three dimensional space, as mentioned in the introduction. Upon the assumption of a plane satisfying the ordinary incidence and order relations, and, furthermore, the parallel axiom, Hilbert* demonstrates that the validity of the Desargues theorem is the necessary and sufficient condition that the plane constitutes a part of a three-dimensional geometry with analogous relations. That the Desargues theorem holds for a plane of such a three space is well known. On the other hand, by the aid of an algebra of segments based upon the Desargues theorem and the parallel axiom, and a resulting analytic geometry, Hilbert exhibited an analytic three-space containing the original plane, while Schor† later exhibited a geo-

* *Grundlagen der Geometrie, Festschrift,* § 29.

† Schor, D., *Neuer Beweis eines Satzes aus den Grundlagen der Geometrie von Hilbert.* Mathematische Annalen, vol. 58 (1904), pp. 427–433.

metric three-space without the algebra of segments. The method here used is analogous to that of Schor, but makes no use of the parallel axiom; this is possible in view of the use of the ideal elements already introduced on the basis of the Desargues theorem. The ideal elements of the resulting three-space, which were not considered by either Hilbert or Schor, are also developed in terms of the original plane system.

Let three special distinct pencils Q_u, Q_v, Q_w of r_∞ be chosen. Ranges S_k distinct from r_∞ and belonging to Q_u, Q_v, Q_w respectively, will be denoted by u_k, v_k, w_k, respectively. A 3-point will then be defined as follows:

Every triple of ranges u_k, v_k, w_k, not in pencil, is a 3-point. Also since every pencil S_k, with the pencils Q_u, Q_v, Q_w, respectively, determines three ranges u_k, v_k, w_k, respectively, which are in pencil, and conversely every three ranges u_k, v_k, w_k which are in pencil determine a pencil S_k, every pencil S_k will also be called a 3-point, and the notation $T_k \equiv u_k, v_k, w_k$ will be used for 3-points of either class. If T_x is a 3-point, u_x and v_x, u_x and w_x, v_x and w_x, determine three pencils W_x, V_x and U_x, respectively, which are coincident or distinct according as T_k is or is not a pencil.

DEFINITION. Three 3-points T_x, T_y, T_z are in the order $T_x T_y T_z$ if and only if the pencils U_x, U_y, U_z are in the order $U_x U_y U_z$ as pencils.

In view of the Lemma to Theorem III, the order $U_x U_y U_z$ implies also the orders $V_x V_y V_z$ and $W_x W_y W_z$, and conversely. Also all three, or any one, of the three 3-points having order relations may be pencils, but if two are pencils the third is a pencil.

If three 3-points T_x, T_y, T_z are collinear, U_x, U_y, U_z are in range, V_x, V_y, V_z are in range, W_x, W_y, W_z are in range, and these three ranges are in pencil Q_{xyz}.

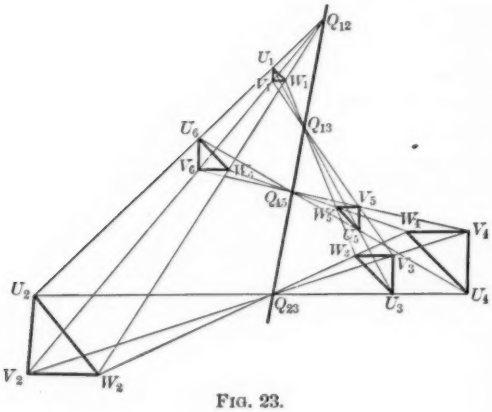

FIG. 23.

It will now be shown that the set of elements T_k satisfies the Axioms I–XI, except IX, and in place of IX possesses a property IX′, characteristic of three-dimensionality.

The validity of Axioms I–VII inclusive follows immediately from our definitions and their validity with respect to the elements S.

AXIOM VIII. (Fig. 23.) *General Case.* If T_1, T_2, T_3 are any three non-collinear 3-points, forming a triangle, and T_4 is a 3-point in the order $T_2 T_3 T_4$, and T_5 is a 3-point in the order $T_3 T_5 T_1$, then there exists a 3-point T_6 in the order $T_1 T_6 T_2$, and on the line $T_4 T_5$.

Proof. Let Q_{12} be the pencil determined by the ranges U_1U_2, V_1V_2, W_1W_2; Q_{45} the pencil determined by the ranges U_4U_5, V_4V_5, W_4W_5; Q_{23} the pencil determined by U_2U_3, V_2V_3, W_2W_3; and Q_{13} the pencil determined by the ranges U_1U_3, V_1V_3, W_1W_3.

Then we have two Desargues configurations.

$$\begin{array}{ccc} & (A) & \\ & Q_\infty & \\ V_1 & V_2 & V_3 \\ U_1 & U_2 & U_3 \\ \hline Q_{23} & Q_{13} & Q_{12} \end{array} \qquad \begin{array}{ccc} & (B) & \\ & Q_\infty & \\ U_3 & U_4 & U_5 \\ V_3 & V_4 & V_5 \\ \hline Q_{45} & Q_{13} & Q_{23} \end{array}$$

Hence from (A) and (B) together we have Q_{23}, Q_{13}, Q_{12}, Q_{45} in range, and this gives us the hypotheses of the Desargues configuration (C), (D), (E).

$$\begin{array}{ccc} & (C) & \\ & Q_{23} & \\ Q_{45} & W_4 & U_4 \\ Q_{12} & W_2 & U_2 \\ \hline Q_v & U_6 & W_6 \end{array} \qquad \begin{array}{ccc} & (D) & \\ & Q_{23} & \\ Q_{45} & W_4 & V_4 \\ Q_{12} & W_2 & V_2 \\ \hline Q_u & V_6 & W_6 \end{array} \qquad \begin{array}{ccc} & (E) & \\ & Q_{23} & \\ Q_{45} & U_4 & V_4 \\ Q_{12} & U_2 & V_2 \\ \hline Q_w & V_6 & U_6 \end{array}$$

From (C), (D), and (E), we know that the set of ranges U_6W_6, U_6V_6, V_6W_6 is a 3-point. It still remains to show that this 3-point is in the order $T_1T_6T_2$. But the figure formed by the pencils Q_u, Q_v, Q_w, U_1, U_2, U_3, U_4, U_5, U_6 is precisely the figure of Axiom VIII, Section III, whence U_6 is in the order $U_1U_6U_2$, and hence T_6 is in the order $T_1T_6T_2$. This also shows that no one of the three pencils U_6, V_6, W_6 could be on the range r_∞.

It is obvious that one or more of the 3-points T_1, T_2, T_3, T_4, T_5, T_6 could be pencils, but the above general proof is easily modified to fit all the special cases, and the details of these proofs are omitted.

Axiom IX is evidently *not* satisfied.

Axiom X. The hypothesis of Axiom X gives at once

$$\begin{array}{ccc} X_{12} & X_{02} & X_{01} \\ \hline X_{40} & X_{41} & X_{42} \\ X_{30} & X_{31} & X_{32} \\ & X^*_{34} & \end{array} \qquad (X = U, V, W),$$

the asterisk indicating that U^*_{34}, V^*_{34}, W^*_{34} may or may not be a 3-point, but simply that U^*_{34}, V^*_{34}, W^*_{34} exist as pencils, whether or not on r_∞.

We have then at once the configurations,

$$\begin{array}{ccc} & Q_z & \\ X_{m4} & X_{m3} & X_{mn} \\ Y_{m4} & Y_{m3} & Y_{mn} \\ \hline Q_{mn3} & Q_{mn4} & Q_{m34} \end{array} \qquad \begin{array}{ccc} & Q_z & \\ X_{n4} & X_{n3} & X_{nm} \\ Y_{n4} & Y_{n3} & Y_{nm} \\ \hline Q_{nm3} & Q_{nm4} & Q_{n34} \end{array}$$

for $z, X, Y = u, V, W; v, W, U; w, U, V$; and $m, n = 1, 2; 0, 1; 0, 2$.

From these configurations we have at once the collinearities $Q_{124}Q_{134}Q_{123}Q_{234}$, $Q_{013}Q_{014}Q_{034}Q_{134}$ and $Q_{023}Q_{024}Q_{034}Q_{234}$. These give the hypotheses for the configurations

$$\begin{array}{ccc} & Q_{jmn} & \\ X_{jm} & Y_{jm} & Q_{34m} \\ X_{jn} & Y_{jn} & Q_{34n} \\ \hline Y^*_{34} & X^*_{34} & Q_z \end{array}$$

for $j = 3, 4$; $m, n = 0, 1$; $0, 2$; $1, 2$; and $X, Y, z = U, V, w; V, W, u; W, U, v$.

From this it follows that $U^*_{34}, V^*_{34}, W^*_{34}$ is a 3-point if these pencils do not lie on r_∞, and if it is not a 3-point they do lie on r_∞.

Now let $T_{31}, T_{41}; T_{30}, T_{40}; T_{32}, T_{42}$ be any pairs whatever of 3-points on the same 3-lines, such that T_{01}, T_{02}, T_{12} exist, and it must be shown that T_{01}, T_{02}, T_{12} are collinear.

We have at once the Desargues configurations

$$\begin{array}{ccc} & X^*_{34} & \\ X_{31} & X_{30} & X_{32} \\ X_{41} & X_{40} & X_{42} \\ \hline X_{02} & X_{12} & X_{01} \end{array} \qquad (X = U, V, W),$$

and then the configurations

$$\begin{array}{ccc} & Q_z & \\ X_{ik} & X_{il} & X_{ij} \\ Y_{ik} & Y_{il} & Y_{ij} \\ \hline Q_{ijl} & Q_{ijk} & Q_{ikl} \end{array}$$

where $i, j, k, l = 0, 1, 2, 3, 4$ but all different and $z, X, Y = u, V, W; v, W, U; w, U, V$. These give the collinearities $Q_{ijk}, Q_{ijl}, Q_{ikl}, Q_{jkl}$ $(i, j, k, l = 0, 1, 2, 3, 4$ but all different).

These give the hypotheses for the configurations

$$\begin{array}{ccc} & Q_i & \\ Q_{ij} & X_{i4} & Y_{i4} \\ Q_{i3j} & X_{i3} & Y_{i3} \\ \hline Q_z & Y_{ij} & X_{ij} \end{array} \quad \left(\begin{matrix} i, j = 0, 1; 0, 2; 1, 2 \\ z, X, Y = \begin{cases} u, V, W \\ v, W, U \\ w, U, V \end{cases} \end{matrix}\right), \qquad \begin{array}{ccc} Q_u & Q_v & Q_w \\ \hline U_{ij} & V_{ij} & W_{ij} \\ U_{ik} & V_{ik} & W_{ik} \\ & Q_{012} & \end{array} \quad \left(\begin{matrix} i, j, k = 0, 1, 2, \\ i \neq j \neq k \neq i \end{matrix}\right),$$

whence T_{01}, T_{02}, and T_{12} are collinear, and we have the conclusion of the theorem.

AXIOM XI. The hypothesis of three 3-points in a range implies the existence of a pair of triangles the joins of whose corresponding vertices are in a pencil.

But, conversely, by Axiom X, this implies that the three 3-points are in a 3-line. But any pair of triangles with sides through these three collinear points are in pencil, by definition. Hence Axiom XI holds.

DEFINITION. A tetrahedron is a set of four planes, determined by four non-coplanar points taken in triples. The existence of such points follows from the fact that Axiom IX does not hold.

DEFINITION. A 3-space is the set of all points collinear with two distinct points of a tetrahedron.

THEOREM. (Fig. 24.) The set of 3-points T constitutes a 3-space.

Proof. Let $T_3 \equiv U_3$, V_3, W_3 (distinct) be any 3-point. Then, since U_3, V_3, W_3 as pencils of the set S, are themselves 3-points, we have a tetrahedron, T_3, U_3, V_3, W_3. Let $T_4 \equiv U_4$, V_4, W_4 be any other 3-point. Then U_3U_4, V_3V_4, W_3W_4, belong to a pencil Q_{34}. If Q_{34} is not a pencil of the range r_∞, it is a 3-point of the plane determined by U_3, V_3, W_3, and T_4 is collinear with T_3 and Q_{34}, and hence is a point of the 3-space $T_3U_3V_3W_3$.

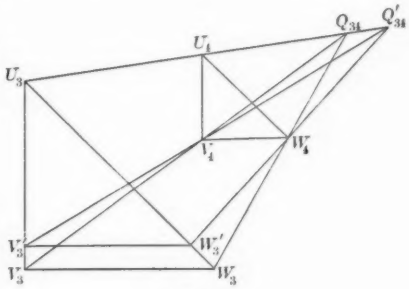

FIG. 24.

If Q_{34} is on r_∞, choose a new pencil V'_3 in the order $V_3V'_3U_3$. Then V'_3Q_u meets U_3Q_v in W'_3, and we have a 3-point T'_3. T'_3 is in the plane determined by $V_3U_3T_3$, and T'_3 and T_4 belong to a pencil $Q'_{34} \neq Q_{34}$, hence not on r_∞. Hence T_4 is collinear with two distinct points of the tetrahedron, and the points T constitute a 3-space.

Ideal elements of the 3-space T. We have now a three-dimensional set of points satisfying order relations, and the ideal 3-space elements might be introduced in it directly, by the methods of Bonola, Schur, or Veblen.* It is interesting, from the point of view of the present paper, to do this in terms of the original plane elements.

Let $T_3 \equiv U_3, V_3, W_3$ and $T_4 \equiv U_4, V_4, W_4$ be any two 3-points. The ranges U_3U_4, V_3V_4, W_3W_4 meet the range r_∞ in three pencils $Q_{u'}$, $Q_{v'}$, $Q_{w'}$, which are in general distinct. The set of three pencils $Q_{u'}$, $Q_{v'}$, $Q_{w'}$, whether or not distinct, will be defined as an *ideal element* of the 3-space T, and will be said to lie on the 3-line $(U_3U_4,\ V_3V_4,\ W_3W_4)$. If these ideal 3-points are adjoined to the set T, giving a set T', the points of T' will not satisfy the order relations as defined, since the set Q did not, but will, evidently, have the following three properties:

Any two planes determine one and only one line.

*Cf. the previous citations.

Any two lines of a plane determine a point.

The general Desargues theorem.

In short, the set of elements T' has all the intersectional properties of a projective 3-space, except those which are dependent on Pascal's theorem. These must be excepted, since Pascal's theorem is independent of the Desargues theorem (cf. Hilbert).

Section V.

Extension to n-Space.

The method of Section IV may now be extended to cover the definition of n-space in terms of the plane set of elements already considered. We choose on the line r_∞ of the set of elements Q, n pencils, $Q_1, Q_2, \cdots, Q_n$. Any set of n ranges $r_i\,(i=1,2,\cdots,n)$, one belonging to each of the pencils $Q_1, \cdots, Q_n$, respectively, and each distinct from the range r_∞, will be defined as an n-point, and denoted by the notation N_x. These n ranges, taken in pairs, determine $n(n-1)/2$ pencils (not necessarily distinct), $S_{x,ij}$ $(i, j=1, \cdots, n;\ i \neq j)$. Let two such n-points be N_x and N_y. The pencils $S_{x,ij}$ and $S_{y,ij}$ $(i,j=1, \cdots, n;\ i \neq j)$ determine a range $r_{xy,ij}$ such that $r_{xy,ij}, r_{xy,ik}, r_{xy,jk}$ $(i,j,k=1, \cdots, n;\ i \neq j \neq k \neq i)$ are in pencil.

The set of $\frac{1}{2}n(n-1)$ ranges $r_{xy,ij}$ will be called an n-line. An n-point N_z will be said to be on the n-line n_{xy}, if and only if, the pencil $S_{z,ij}$ belongs to the range $r_{xy,ij}\,(i,j=1, \cdots, n;\ i \neq j)$. Its *order* will be said to be the same as the order of the pencils $S_{x,ij}$, $S_{y,ij}$, $S_{z,ij}$. That this definition of order is unique is easily seen as follows. Each i, j, k may be thought of as fixing a 3-space geometry in which the order for ij, ik, jk is unique, and by a sequential process the result holds for all of the ranges, and hence we have the general result for the n-space. Hence we may write the order $N_x N_y N_z$ as implying the orders $S_{x,ij} S_{y,ij} S_{z,ij}$ for each triple of values for i, j, k, and conversely. The n-point N_x will be said to be the same as the pencil S_x if $S_{x,ij}$ is S_x for every ij, since S_x will in this case completely determine the n-point.

It can now be shown that this n-point geometry satisfies all the axioms except that of closure. Axioms I–VII inclusive follow immediately from the definitions. The validity of Axiom VIII may be shown as follows. The axiom states that N_1, N_2, N_3 are the vertices of a triangle, and if N_4 and N_5 are in the orders $N_2 N_3 N_4$ and $N_1 N_5 N_3$ respectively, then there exists an n-point N_6 in the order $N_1 N_6 N_2$ and on the line $N_4 N_5$. Consider any triple of the pencils $Q_1, \cdots, Q_n$, of r_∞, say Q_i, Q_j, Q_k. Then we have the existence of the three ranges $r_{6,i}$, $r_{6,j}$, $r_{6,k}$, of the n-point, N_6, from the corresponding theorem of the 3-space theory. Similarly, if i, j, k be allowed to take each set of values (distinct and of the set $1 \cdots n$), we obtain all of the ranges of the set forming the n-point N_6, and each of the pencils $S_{6,ij}$ will be in the order required from the corresponding 3-space theorem. Hence Axiom VIII is valid.

That Axiom IX does not hold is evident, since each pencil S_x is an n-point, but there are other n-points which do not lie in the plane of the pencils Q.

Axioms X and XI may be easily shown to hold by considering the pencils $Q_1, \cdots, Q_n$ of the range r_∞ in sets of triples, and then applying the theorem already proved for each of the resulting 3-spaces, thus obtaining the theorem for n-space.

That the space thus obtained is actually an n-space may be shown as follows: By the same method of proof as in Section IV, if for $Q_1, \cdots, Q_n$, we have an n-space, then by increasing the number of pencils by adding Q_{n+1} to the set, we introduce new elements into the geometry. But these may all be seen to be collinear with two points of $n+1$ n-spaces, and hence constitute an $(n+1)$-space. But the theorem is true for 3-space, and hence by induction, for any value of n.

The ideal elements which must be adjoined to this n-space to make it projective are the pencils (n in number, but not necessarily distinct) belonging to the range r_∞, in which any other n-line (i. e., its component ranges) meet this range r_∞. If we call each such set an ideal n-point, and adjoin all such ideal points to the set already obtained, we have a set possessing the properties that any two n-points determine an n-line, any two n-lines determine an n-point, and the general Desargues Theorem. The first two of these statements are evident, and the last may readily be seen by considering the parts of the figure separately, as in the proof of Axiom VIII.

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THE GROUP OF CLASSES OF CONGRUENT QUADRATIC INTEGERS WITH RESPECT TO A COMPOSITE IDEAL MODULUS*

BY

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Introduction.

If in the ordinary theory of rational numbers we consider a composite integer m as modulus, and if from among the classes of congruent integers with respect to that modulus we select those which are prime to the modulus, they form a well-known multiplicative group, which has been called by WEBER (*Algebra*, vol. 2, 2d edition, p. 60), the most important example of a finite abelian group. In the more general theory of numbers in an algebraic field we may in a corresponding manner take as modulus a composite ideal, which includes as a special case a composite principal ideal, that is, an integer in the field, and if we regard all those integers of the field which are congruent to one another with respect to the modulus as forming a class, and if we select those classes whose integers are prime to the modulus, they also will form a finite abelian group† under multiplication.

The investigation of the nature of this group is the object of the present paper. I shall confine my attention, however, to a quadratic number-field, and shall determine the structure of the group of classes of congruent quadratic integers with respect to any composite ideal modulus whatever. Several distinct cases arise depending on the nature of the prime ideal factors of the modulus; for every case I shall find a complete system of independent generators of the group.

Exactly as in the simpler theory of rational numbers it will appear that the solution of the problem depends essentially on the case in which the modulus is a prime-power ideal, that is, a power of a prime ideal. The most important case, however, is probably that in which the modulus is a rational principal ideal or in other words a rational integer; therefore a separate discussion will be given of this case. Another interesting case is that in which the group is

* Presented to the Society (Chicago), April 9, 1909.

† This group must not be confused with the group of classes of equivalent ideals in an algebraic field, which is also finite and abelian; the latter has been the subject of numerous investigations, in which, however, the *language* of the theory of groups has not usually been employed.

cyclic and primitive roots exist; this case arises in a variety of ways, all of which will be enumerated, and the corresponding primitive roots given.

Preliminary notions.

1. Let m be any rational integer whose prime factors are all distinct, and let $\omega = \sqrt{m}$, if $m \equiv 2$ or $3 \pmod 4$, and $\omega = \frac{1}{2}(1 + \sqrt{m})$, if $m \equiv 1 \pmod 4$; then in the quadratic number-field defined by $\sqrt{m}$ the numbers $a + b\omega$, where a and b are rational integers, are said to be the *integers of the field*, or simply *quadratic integers*. The factorization of quadratic integers, although apparently capricious and utterly lawless, has been shown to be really amenable to reason, when considered from the standpoint of, and in connexion with, certain artificial quantities called *ideals*. For an elementary and detailed discussion of ideals in a quadratic number-field, illustrated by concrete examples, see SOMMER'S recent book *Vorlesungen über Zahlentheorie*,* especially pages 36–59.

Unless otherwise stated, the notation and nomenclature used by SOMMER will be followed in this paper. Thus rational integers will be denoted by Roman letters a, b, etc., quadratic integers by Greek letters α, β, etc., and *principal ideals* (Hauptideale) by the symbols (a), (α), etc. But ideals in general will be denoted by capitals A, P, etc., except that the particular letters G, H, J will be reserved for groups.

2. Let A be any ideal of the number-field (Zahlkörper) $\mathrm{k}(\sqrt{m})$; the number of integers of the field that form a complete system of residues with respect to the modulus A, that is, the number of classes of congruent integers with respect to A, is equal to the norm of A, $n(A)$. Among these classes those which are prime to A, $\Phi(A)$ in number, where the Φ-function is a generalization of Euler's ϕ-function,† evidently form an *abelian multiplicative group* G, of order $\Phi(A)$. However, it will be more convenient to let the group be made up, not of the classes themselves, but of $\Phi(A)$ quadratic integers, one chosen from each class. In other words we shall select from a complete system of residues those which are prime to the modulus, thus forming what will be called a *reduced system of residues*, and regard them as the elements of the group. This is legitimate, provided we agree that the product of two elements α and β of the group is congruent to a third element γ of the group, $\alpha\beta \equiv \gamma \pmod A$, and not necessarily equal to it. The rational integer 1 can always be taken to be the identical element of the group.

* The reader of this book should be on the lookout for a number of minor errors and some rather misleading statements. Reference will also be made to HILBERT'S *Bericht* in the Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 4 (1897), pages 181–194, for the source of much of the material in SOMMER'S book, and to BACHMANN'S *Neuere Zahlentheorie*, Sammlung Schubert, for an introduction to the theory of algebraic numbers, in which the author, like SOMMER, confines himself to the quadratic field.

† SOMMER, l. c., pp. 78–81.

Modulus a Power of a Prime Ideal.

3. In studying this group we shall first take up the case in which the modulus A is a power of a prime ideal P, say $A = P^k$. If A is itself prime ($k = 1$), it is known that the group is *cyclic*, and therefore that *primitive roots* exist with respect to a prime ideal P (Sommer, l. c., pp. 84–86).

In order to investigate the case $k > 1$, it will be necessary to consider separately the three kinds of prime ideals, those of the first grade, both unambiguous and ambiguous, and those of the second grade (Sommer, l. c., p. 48 and p. 53). Let p be a rational prime such that the principal ideal (p) is divisible by the prime ideal P, and let P' be the conjugate of P; then the three kinds of prime ideals may be briefly characterized as follows:

First grade; unambiguous, $PP' = (p)$, $P' \neq P$,

ambiguous, $P^2 = (p)$.

Second grade; $P = (p)$.

Again, if P is of the first grade, $n(P^k) = p^k$ and $\Phi(P^k) = p^{k-1}(p-1)$, and if P is of the second grade, $n(P^k) = p^{2k}$ and $\Phi(P^k) = p^{2k-2}(p^2-1)$.

It will also be necessary to distinguish between different values of the rational prime p, according as $p = 2$, $p = 3$, or $p > 3$, and again between different forms of m, according as $m \equiv 1, 2$, or $3 \pmod 4$.

From the canonical forms of the bases of the different kinds of prime ideals, as given by Hilbert (*Bericht*, pp. 282–283) and Sommer (pp. 59–63), the corresponding canonical forms of the bases of their kth powers are easily derived and will be assumed in what follows.

Case I. *P an Unambiguous Prime of the First Grade.*

4. In this case $P = (p, a + \omega)$ and $P^k = (p^k, a_k + \omega)$; since the coefficient of ω in the basis of the modulus is 1, every integer in the field is congruent to some rational integer, and we may choose as a complete system of residues with respect to the modulus the rational integers $0, 1, \cdots, p^k - 1$. Now those which are prime to P^k are precisely those which are prime to p; so if we strike out from this complete system of residues those integers which are divisible by p, we shall have left $p^{k-1}(p-1)$ residues, one from each of the classes forming the group in question. In this first case, therefore, the group is simply isomorphic with the corresponding group in the rational number-field, and its generators can be written down immediately. There are two types, depending on the value of p.

Type 1. $p > 2$.

5. Thus, if $p > 2$, G is a cyclic group and can be generated by a single element of period $p^{k-1}(p-1)$. But it can also be generated by two indepen-

dent elements, one of period $p-1$ and the other of period p^{k-1}; and the latter choice of generators will be found more convenient for our purpose. Throughout this paper, similarly, the periods of the generators will always be chosen in such a way that no period shall be divisible both by p and by another prime factor.

Let us take $1+p$ as generator of period p^{k-1}, if $k>1$; if $k=1$, it reduces to the identical element and no longer figures as a generator. To find a generator of period $p-1$, let f_1 be a primitive root of the rational prime p; with respect to the modulus P^k, f_1 will be of period $p^i(p-1)$, where $i<k$, and $f=f_1^{p^i}$ will be of period $p-1$. *Therefore we shall take f and $1+p$ as the two required independent generators, of periods $p-1$ and p^{k-1}, respectively.*

Type 2. $p=2$.

6. If $p=2$, we obtain the second type, which is a non-cyclic group of order 2^{k-1}, having the invariants $2, 2^{k-2}(k>2)$. *Its generators will be taken to be -1 of period 2, and 5 of period 2^{k-2}. If $k=2$, there is only one generator, -1 of period 2; and if $k=1$, the group is of order 1.*

Case II. *P an Ambiguous Prime Ideal.*

7. In this case P is necessarily of the first grade and its norm p is a factor of the discriminant d of the number-field. Since $P^2=(p)$, every even power of P is a rational principal ideal, $P^{2s}=(p^s)$, and every odd power is equal to P multiplied by a rational principal ideal, $P^{2s+1}=(p^s)\cdot P$. This case will be found to give rise to five distinct types of groups, in two of which $p>2$ and in the other three $p=2$.

Subcase II_a. $p>2$.

8. By reference to Sommer's text, pages 59–63, we see that if $m\equiv 2$ or $3 \pmod 4$, $P=(p, \sqrt{m})$, and that if $m\equiv 1 \pmod 4$, then

$$\omega=\frac{1+\sqrt{m}}{2}, \qquad P=\left(p, \frac{p-1}{2}+\omega\right)=\left(p, \frac{p+\sqrt{m}}{2}\right).$$

It will be convenient to combine these two cases into one by introducing the symbol ω_1, defined by the equations

$$\omega_1=\omega=\sqrt{m}, \qquad \text{if } m\equiv 2 \text{ or } 3 \pmod 4,$$

$$\omega_1=\frac{p-1}{2}+\omega=\frac{p+\sqrt{m}}{2}, \qquad \text{if } m\equiv 1 \pmod 4.$$

In both cases, therefore, we may write

$$P=(p, \omega_1).$$

9. Further, raising both sides of this equation to even and to odd powe s, respectively, we have

$$(1) \qquad P^{2s} = (p^s, p^s\omega_1), \qquad P^{2s+1} = (p^{s+1}, p^s\omega_1).$$

In order to obtain a notation by means of which the even and the odd powers of P can be considered together, we again introduce new symbols, r and s, defined by the equations

$$\left[\frac{k}{2}\right] = s, \qquad \left[\frac{k+1}{2}\right] = r,$$

where k is a positive rational integer, and $[x]$ means the largest rational integer $\leqq x$. It follows that if $k = 2s$, $r = s$, and if $k = 2s+1$, $r = s+1$, and in either case $k = r + s$. By means of these symbols the two equations (1) can be replaced by the single equation

$$P^k = (p^r, p^s\omega_1).$$

10. Since $m \equiv 0 \pmod{p}$ and (ω_1) is not prime to (p), we can select as a reduced system of residues with respect to the modulus P^k the quadratic integers $u + v\omega_1$, in which v takes all rational integral values from 0 to $p^s - 1$, inclusive, and u takes all rational integral values from 1 to $p^r - 1$, inclusive, except those which are divisible by p. This gives p^s values to v and $p^{r-1}(p-1)$ values to u, and therefore furnishes $p^{r+s-1}(p-1)$ residues for the reduced system. This agrees with the formula $\Phi(P^k) = p^{k-1}(p-1)$ for the order of the group, since $r + s = k$.

A more convenient choice of residues for our purpose, however, is $u + v\sqrt{m}$, where u and v have the same values as before. If $m \equiv 2$ or $3 \pmod 4$, $\omega_1 = \sqrt{m}$ and the residues are also the same as before; if $m \equiv 1 \pmod 4$, $\omega_1 = \frac{1}{2}(p + \sqrt{m})$ and the residues are different, but still form a reduced system, as is evident from the formulæ

$$\sqrt{m} = -p + 2\omega_1,$$

$$\omega_1 \equiv -p\frac{p^{r-1}-1}{2} - \frac{p^s-1}{2}\sqrt{m} \pmod{P^k}.$$

11. Under this subcase all the groups are, for $p > 3$, of a single type, the third; while if $p = 3$, another type, the fourth, also occurs. In either type, however, it is immediately evident that the quadratic integers having rational residues form a cyclic subgroup analogous to type 1, and generated, therefore, by two rational integers, viz., g, of period $p - 1$, and (if $r > 1$) $1 + p$, of period p^{r-1}. If $p = 3$, $g = -1$.

12. Among the irrational elements of the group (if $k > 1$) the integer $1 + \sqrt{m}$ suggests itself as probably an available independent generator of period p^r. This conjecture will be found to be correct, if $p > 3$, and sometimes, but not always,

correct, if $p = 3$. Here the word *independent* means *independent of the rational residues.*

THEOREM. *If $k > 1, 1 + \sqrt{m}$ is an independent generator of period p^s with respect to the modulus $P^k = (p^r, p^s\omega_1)$ in every case except that in which $p = 3$ and $m \equiv -3 \pmod 9$.*

Proof. By means of the binomial theorem we obtain the formula

$$(2)\quad (1+\sqrt{m})^{p^i} \equiv 1 + p^{i+1}\cdot\frac{p^i-1}{2}\cdot\frac{m}{p} + p^i\left[1 + \frac{p(p^i-1)(p^i-2)}{2\cdot 3}\cdot\frac{m}{p}\right]\sqrt{m} \quad (\text{modd } p^{i+2},\ p^{i+2}\sqrt{m}),$$

where the coefficients are apparently rational fractions, but really integers. Putting $i = s$ in this congruence, and noting that $i + 1 \geqq r$, we see that $(1 + \sqrt{m})^{p^s} \equiv 1 \pmod{P^k}$ and therefore that $1 + \sqrt{m}$ has a period which is a divisor of p^s and will have a period exactly equal to p^s and will be independent of the rational residues, if its (p^{s-1})th power is irrational. Putting $i = s - 1$ in formula (2), we see that

$$(1+\sqrt{m})^{p^{s-1}} \equiv 1 + p^s\cdot\frac{p^{s-1}-1}{2}\cdot\frac{m}{p} + p^{s-1}\cdot y\cdot\sqrt{m} \quad (\text{modd } p^{s+1},\ p^{s+1}\sqrt{m}),$$

where

$$y = 1 + \frac{p(p^{s-1}-1)(p^{s-1}-2)}{2\cdot 3}\cdot\frac{m}{p}.$$

Hence $(1 + \sqrt{m})^{p^{s-1}}$ will have an irrational residue with respect to the modulus $(p^r, p^s\omega_1)$, if $y \not\equiv 0 \pmod p$. This is the case when $p > 3$; for then $y \equiv 1 \pmod p$: it is also the case, when $p = 3$ and $m \equiv 3 \pmod 9$; for then $y \equiv 1 + \frac{1}{2}m \equiv 2 \pmod 3$. The only other possibility, since $p > 2$, is $p = 3$ and $m \equiv -3 \pmod 9$. But this supposition makes $y \equiv 0 \pmod 3$, and the argument fails. The theorem is therefore proved.

Type 3. $p > 3$, *or* $p = 3$ *and* $m \equiv 3 \pmod 9$.

13. This theorem enables us to complete the determination of the third type of group. *Thus if P is an ambiguous prime ideal and if either $p > 3$ or $p = 3$ and $m \equiv 3 \pmod 9$, then the group of reduced residues with respect to the modulus $P^k = (p^r, p^s\omega_1)$ is generated by the three independent elements g, $1 + \sqrt{m}$* * *(if $k > 1$), and $1 + p$ (if $k > 2$), whose periods are $p - 1$, p^s, and p^{r-1}, respectively.*

In other words every quadratic integer α, prime to the modulus, is expressible in one and only one way as a product of the form

$$\alpha \equiv g^{n_1}(1+\sqrt{m})^{n_2}(1+p)^{n_3} \pmod{P^k},$$

* It is to be noticed that if $m \equiv 1 \pmod 4$, $1 + \sqrt{m} = 2\omega$.

where the exponents n_1, n_2, and n_3 are residues with respect to the corresponding periods $p-1$, p^s, and p^{r-1}, respectively, and will be called the *indices* of α. If these indices range independently over complete systems of residues as to their respective moduli, the quadratic integer α will range over the entire group G. A similar definition of indices applies, of course, to every type of group.

14. It is evident that the elements whose periods are powers of p all have residues of the form $1+pu+v\sqrt{m}$ and constitute a Sylow subgroup H of order p^{k-1}, generated by $1+\sqrt{m}$ and $1+p$. It may easily be verified that the system of residues of the form $1+p^iu+p^jv\sqrt{m}$, where i and j are fixed exponents such that $1\leqq i\leqq r$ and $0\leqq j\leqq s$, constitute a group (a subgroup of H of order p^{k-i-j}), if, and only if, $i\leqq 2j+1$. Again, an element α is of period $p^i\,(i\leqq s)$, if, and only if, it is expressible in the form

$$\alpha\equiv 1+p^{r-i}u+p^{s-i}v\sqrt{m}\pmod{P^k},$$

where u and v are not both divisible by p.

Type 4. $p=3$, $m\equiv -3 \pmod 9$.

15. We now come to the exceptional case in which $p=3$, $m\equiv -3 \pmod 9$, and therefore $P^k=(3^r,\ 3^s\omega_1)$; it gives rise to another type of group, the fourth; $1+\sqrt{m}$ is no longer available as a generator, because it is not, in general, independent of the rational residues.

Let H, as before, be the subgroup of G whose elements are of the form $1+3u+v\sqrt{m}$ and whose order is 3^{k-1}, and let $J\,(k>1)$ be the subgroup of H whose elements are of the form $1+3u+3v\sqrt{m}$ and whose order is 3^{k-2}. From the formula

$$(1+3\sqrt{m})^{3^i}\equiv 1+3^{i+1}\sqrt{m}\pmod{3^{i+3},\ 3^{i+3}\sqrt{m}}$$

it is immediately clear that $1+3\sqrt{m}$ is of period 3^{s-1} and independent of the rational residues, and consequently that if $k>3$, J has two independent generators, which can be taken to be 4 and $1+3\sqrt{m}$, of periods 3^{r-1} and 3^{s-1}, respectively.

16. We wish to show that, if $k>3$, H has the invariants $(3^{r-1},\ 3^{s-1},\ 3)$ and three independent generators, including those of J and one additional generator λ of period 3. This conclusion can be drawn the moment it is proved that H contains an element of period 3 not contained in J, that is, an element

$$\lambda=x+y\sqrt{m}, \tag{3}$$

such that

$$\lambda^3\equiv 1\pmod{P^k} \tag{4}$$

and

$$y\not\equiv 0\pmod 3. \tag{5}$$

It is unnecessary to write $1+3u$ for x; the conditions (4) and (5) are sufficient to show that λ occurs in H. From the abstract properties of the group H we know that if there exists one such element λ, *there must exist exactly eighteen*, any one of which will suffice as an independent generator of H and therefore also of G.

17. THEOREM. *If x and y are rational integers satisfying the congruences*

$$2x \equiv -1 \pmod{3^{r-1}}, \qquad r>1, \tag{6}$$

$$4\cdot\frac{m}{3}\cdot y^2 \equiv -1 \pmod{3^{s-1}}, \qquad s>1, \tag{7}$$

then $\lambda = x + y\sqrt{m}$ will satisfy conditions (4) *and* (5).

Proof. (5) is an immediate consequence of (7). To show that (4) is also satisfied, we use the formula

$$8\lambda^3 = (2x)^3 + 9\cdot 2x\cdot 4\cdot\frac{m}{3}\cdot y^2 + 6y\left(4x^2 + 4\frac{m}{3}y^2\right)\sqrt{m},$$

from which, in view of (6) and (7) and the congruence $(2x)^3 \equiv -1 \pmod{3^r}$, derived from (6), we obtain the result

$$8\lambda^3 \equiv -1 + 9(-1)(-1) + 6y(1-1)\sqrt{m} \equiv 8 \quad (\text{modd } 3^r, 3^s\sqrt{m}),$$

and therefore also (4).

18. Since in (6) and (7) the coefficients of x and y are prime to 3, and since in (7) $4m/3$ and -1 are both non-squares, mod 3, therefore these two congruences can always be solved for x and y, and λ exists. Moreover, they have one and two solutions, respectively, which give three and six incongruent solutions, with respect to the moduli 3^r and 3^s, and determine eighteen distinct values of λ of the form $(x \pm y\sqrt{m}) + (3^{r-1}u + 3^{s-1}v\sqrt{m})$, where $u, v = 0, 1, 2$. These are precisely the eighteen residues of period 3 not contained in J. Hence (6) and (7) are necessary as well as sufficient conditions that $\lambda = x + y\sqrt{m}$ be an independent generator of period 3.

19. In order to extend the application of the theorem to the cases $k=2$ and $k=3$, in which $s=1$, we shall restrict x and y, in those cases, to the value 1. This will agree with the obvious fact that $1+\sqrt{m}$ is an independent generator of period 3 with respect to the moduli $(3, 3\omega_1)$ and $(9, 3\omega_1)$. Moreover, when the modulus is $(9, 9\omega_1)$, congruences (6) and (7) show that $1+\sqrt{m}$ is again a generator of period 3; but when the modulus is $(27, 9\omega_1)$, $4+\sqrt{m}$ is the corresponding generator. It is to be noticed that the solution of (6) can be written $x \equiv 1 + 3 + \cdots + 3^{r-2} \pmod{3^{r-1}}$.

There is one case in which the generator of period 3 is a rational multiple of $1+\sqrt{m}$ for all values of k, and that is when $m=-3$; in that case (7) takes the form $(2y)^2 \equiv 1 \pmod{3^{s-1}}$ and is satisfied by $y=x$; therefore $\lambda = x(1+\sqrt{m})$, where x is a root of the congruence (6).

20. The results obtained for type 4 may be summarized as follows: *If* $m \equiv -3 \pmod 9$ *and if* $P^k = (3^r, 3^s\omega_1)$ *is a power of an ambiguous prime ideal whose norm is* 3, *then the group of reduced residues, mod* P^k, *is generated by* -1 *of period* 2, *if* $k = 1$; *if* $k > 1$, *the quadratic integer* λ *determined by* (3), (6), *and* (7), *is an independent generator of period* 3; *if* $k > 2$, 4 *is another independent generator of period* 3^{r-1}; *and if* $k > 3$, $1 + 3\sqrt{m}$ *is still another, and the last, independent generator of period* 3^{s-1}.

The third power of every element of H is obviously equal to the third power of some element of J and is therefore of the form $1 + 9(u + v\sqrt{m})$. More generally, the 3^i-th power of every element of H is of the form $1 + 3^{i+1}(u + v\sqrt{m})$.

21. *Examples.* To illustrate the difference between the third and fourth types, take first the case where $m = -6$, $p = 3$, and $A = P^4 = (9, 9\sqrt{-6})$; here G is of type 3 and is generated by -1 of period 2, 4 of period 3, and $\alpha = 1 + \sqrt{-6}$ of period 9. The cyclic subgroup of order 9 generated by α may be written out as follows:

$$\begin{array}{lll} 1 = 1, & \alpha^3 \equiv 1 + 6\sqrt{-6}, & \alpha^6 \equiv 1 + 3\sqrt{-6}, \\ \alpha = 1 + \sqrt{-6}, & \alpha^4 \equiv 1 + 7\sqrt{-6}, & \alpha^7 \equiv 1 + 4\sqrt{-6}, \pmod{P^4}. \\ \alpha^2 \equiv 4 + 2\sqrt{-6}, & \alpha^5 \equiv 4 + 8\sqrt{-6}, & \alpha^8 \equiv 4 + 5\sqrt{-6}, \end{array}$$

Then take the case where $m = 6$, $p = 3$, and $A = P^4 = (9, 9\sqrt{6})$; G is of type 4 and is generated by -1 of period 2, 4 of period 3, $\lambda = 1 + \sqrt{6}$ of period 3, and $\beta = 1 + 3\sqrt{6}$ of period 3. The non-cyclic subgroup of order 9 generated by λ and β may be written out as follows:

$$\begin{array}{lll} 1 = 1, & \beta = 1 + 3\sqrt{6}, & \beta^2 \equiv 1 + 6\sqrt{6}, \\ \lambda = 1 + \sqrt{6}, & \beta\lambda \equiv 1 + 4\sqrt{6}, & \beta^2\lambda \equiv 1 + 7\sqrt{6}, \pmod{P^4}. \\ \lambda^2 \equiv 7 + 2\sqrt{6}, & \beta\lambda^2 \equiv 7 + 5\sqrt{6}, & \beta^2\lambda^2 \equiv 7 + 8\sqrt{6}, \end{array}$$

Subcase II_b. $p = 2$.

22. Here we shall find three new types of groups, the fifth, sixth, and seventh, depending on the value of m. If $m \equiv 2 \pmod 4$, $P = (2, \sqrt{m})$, $A = P^k = (2^r, 2^s\sqrt{m})$, and G is of the fifth type; while if $m \equiv 3 \pmod 4$, $P = (2, 1 + \sqrt{m})$, $A = P^k = (2^r, 2^s + 2^s\sqrt{m})$, and G is of the sixth or seventh type according as $m \equiv 3$ or $7 \pmod 8$; m cannot be $\equiv 1 \pmod 4$, since in that case p would not be a factor of the discriminant d. In all three types the rational residues form a subgroup J of order 2^{r-1} generated by -1, of period 2, and (if $r > 2$) 5, of period 2^{r-2}. In all three types, also, the case $k = 1$ is trivial and will be left out of account, since G is then of order 1.

Type 5. $m \equiv 2 \pmod 4$.

23. In this case the modulus is $P^k = (2^r, 2^s\sqrt{m})$, and since m is even, the elements of G are the residues $u + v\sqrt{m}$, where u takes the odd values $1, 3, \cdots, 2^r - 1$, and v ranges over *all* the integral values from 0 to $2^s - 1$. The group is easily disposed of, because only one irrational generator, $1 + \sqrt{m}$, is needed. The latter is of period 2^s and independent of the rational residues, exactly as in type 3 (§ 12, Theorem). For in the formula

$$(1 + \sqrt{m})^{2^i} \equiv 1 + 2^i\sqrt{m} \pmod{2^{i+1}, 2^{i+1}\sqrt{m}},$$

which is easily verified, we see, by putting $i = s$ and $s - 1$ in turn, that the 2^s-th power of $1 + \sqrt{m}$ is $\equiv 1 \pmod{P^k}$ and that the 2^{s-1}-th power is irrational.

Therefore the independent generators of G can be taken to be $1 + \sqrt{m}$ *(if* $k > 1$*),* -1 *(if* $k > 2$*), and* 5 *(if* $k > 4$*), and their periods are* $2^s, 2$, *and* 2^{r-2}, *respectively.*

24. On the other hand, if $m \equiv 3 \pmod 4$, the modulus is $P^k = (2^r, 2^s + 2^s\sqrt{m})$.

In both the resulting types of groups the elements are of the form $u + v(1 + \sqrt{m})$ in which u is odd, or in other words they are of the form $u + v\sqrt{m}$ in which one of the coefficients u, v is odd and the other even. In both types there is the same subgroup J formed by the rational residues, and also the same independent irrational generator of period 2^{s-1}, namely $1 + 2\sqrt{m}$. For by expanding in powers of $1 + \sqrt{m}$ and noting that

$$(1 + \sqrt{m})^2 = 2\left(\frac{m-1}{2} + (1 + \sqrt{m})\right),$$

we easily derive the formula

$$(1 + 2\sqrt{m})^{2^i} = [-1 + 2(1 + \sqrt{m})]^{2^i} \equiv 1 - 2^{i+1}(1 + \sqrt{m}) \pmod{2^{i+2}, 2^{i+2}(1 + \sqrt{m})};$$

and by putting $i = s - 1$ and $s - 2$ in turn, we infer that the 2^{s-1}-th power of $1 + 2\sqrt{m}$ is $\equiv 1 \pmod A$ and that the 2^{s-2}-th power is irrational.

So far we have accounted for just half the group G; for J and $1 + 2\sqrt{m}$ generate a subgroup H of index 2 formed by those residues $u + v\sqrt{m}$ for which u is odd and v even. The other half of the group consists of the residues for which u is even and v odd. It will now be necessary to distinguish between the two cases $m \equiv 3$ and $m \equiv 7 \pmod 8$.

Type 6. $m \equiv 3 \pmod 8$.

25. Recalling the well-known fact that in the rational subgroup J the elements which are $\equiv \pm 3 \pmod 8$ are all of period 2^{r-2} $(r > 2)$, while those which are $\equiv \pm 1 \pmod 8$ are all of lower period $(r > 3)$, we see that in this case m is of

period $2^{r-2}(r>2)$ and that J is generated by -1 and m. Now $\sqrt{m}$ is an element of G, but not of H, whose square is m. It is therefore of period $2^{r-1}(r>2)$ and independent of -1; since its even powers are rational, it is also independent of $1+2\sqrt{m}$. Consequently, if $r>2$, we may choose as three independent generators of the group G, $\sqrt{m}$, $1+2\sqrt{m}$, and -1, whose periods are 2^{r-1}, 2^{s-1}, and 2, respectively.

This conclusion requires a slight modification to cover the cases $r=1$ and $r=2$. It is easy to verify the following: if $k=2$ $(r=1, s=1)$, G is generated by $\sqrt{m}$, of period 2; if $k=3$ $(r=2, s=1)$, G is generated by $\sqrt{m}$, of period 4; if $k=4$ $(r=2, s=2)$, G is generated by $\sqrt{m}$, of period 4, and $1+2\sqrt{m}$, of period 2. Every case will be included in the statement that *$\sqrt{m}$ is an independent generator of G, whose period is 2^r, if $k=2$, 3, or 4, and 2^{r-1}, if $k>4$; if $k>3$, $1+2\sqrt{m}$ is another independent generator, of period 2^{r-1}; and if $k>4$, -1 is still another independent generator, of period 2.*

Type 7. $m \equiv 7 \pmod 8$.

26. In this case the period of m is $<2^{r-2}$ $(r>3)$ and that of $\sqrt{m}$ is $<2^{r-1}$. But it is easy to find an element μ of period 4, not contained in H, which, together with 5 and $1+2\sqrt{m}$, will generate G. If one such element exists, *there must be exactly thirty-two*, if $r>2$, since the invariants of G are 2^{r-2}, 2^{r-1}, and 4.

It will be sufficient to put

$$\mu = x\sqrt{m} \tag{8}$$

and determine x as a rational integer satisfying the congruence

$$mx^2 \equiv -1 \pmod{2^{r-1}}. \tag{9}$$

For since $\mu^2 \equiv -1 \pmod{2^{r-1}}$, therefore $\mu^4 \equiv 1 \pmod{2^r}$, and μ is of period 4; since the powers of 5 are all congruent to 1 (mod 8), μ is independent of 5; and since μ^2 is rational, μ is independent of $1+2\sqrt{m}$. Moreover, since $m \equiv -1 \pmod 8$, the congruence $mx^2 \equiv -1 \pmod 8$ and therefore also the congruence (9), can always be solved, and the element μ can always be found.

27. If $r=2$, the congruence (9) evidently still holds and determines the generator $\sqrt{m}$, of period 4. But if $r=1$ and $k=2$, it will be necessary to define $x=1$ as the corresponding root of (9) in order to obtain the generator $\sqrt{m}$; and moreover, its period in this case is 2 instead of 4.

Our final result is, therefore, that *the quadratic integer μ, defined by* (8) *and* (9), *is an independent generator whose period is* 2, *if $k=2$, and* 4, *if $k>2$; if $k>3$, $1+2\sqrt{m}$ is another independent generator, of period 2^{r-1}; and if $k>4$, 5 is still another independent generator, of period 2^{r-2}.*

If $k<9$, and therefore $r<5$, the simplest solution of (9) is $x=1$ and the

simplest choice of the generator μ is $\sqrt{m}$. If $k \geqq 9$, this simple form is no longer in general available; but there is one case in which it is available for every value of k, and that is when $m = -1$. In that case the elements $\sqrt{-1}$, $1 + 2\sqrt{-1}$, and 5 always generate the group.

28. *Examples.* To illustrate types 5, 6, and 7, let m be chosen to be 2, 3, and -1, respectively. If $m = 2$ and $k = 5$, so that $P^5 = (8, 4\sqrt{2})$, then G is of type 5 and order 16 and is generated by $\alpha = 1 + \sqrt{2}$, of period 4, $b = -1$, of period 2, and $c = 5$, of period 2. Thus:

$$\begin{array}{llll} 1 = 1, & \alpha = 1 + \sqrt{2}, & \alpha^2 \equiv 3 + 2\sqrt{2}, & \alpha^3 \equiv 7 + \sqrt{2}, \\ c = 5, & \alpha c \equiv +5\sqrt{2}, & \alpha^2 c \equiv 7 + 2\sqrt{2}, & \alpha^3 c \equiv 3 + \sqrt{2}, \\ b = 7, & \alpha b \equiv 7 + 3\sqrt{2}, & \alpha^2 b \equiv 5 + 2\sqrt{2}, & \alpha^3 b \equiv 1 + 3\sqrt{2}, \\ bc \equiv 3, & \alpha bc \equiv 3 + 3\sqrt{2}, & \alpha^2 bc \equiv 1 + 2\sqrt{2}, & \alpha^3 bc \equiv 5 + 2\sqrt{2}, \end{array} \pmod{P^5}.$$

By way of contrast to this consider a corresponding group of type 7. E. g., take $m = -1$, $k = 5$, and $P^5 = (8, 4 + 4i)$, where $i = \sqrt{-1}$; then G is of order 16 and is generated by $\alpha = i$, of period 4, $\beta = 1 + 2i$, of period 2, and $c = 5$, of period 2. Thus:

$$\begin{array}{llll} 1 = 1, & \alpha = i, & \alpha^2 \equiv 7, & \alpha^3 \equiv 4 + 3i, \\ \beta = 1 + 2i, & \alpha\beta \equiv 6 + i, & \alpha^2\beta \equiv 3 + 2i, & \alpha^3\beta \equiv 6 + 3i, \\ c = 5, & \alpha c \equiv 4 + i, & \alpha^2 c \equiv 3, & \alpha^3 c \equiv 3i, \\ \beta c \equiv 5 + 2i, & \alpha\beta c \equiv 2 + i, & \alpha^2\beta c \equiv 7 + 2i, & \alpha^3\beta c \equiv 2 + 3i, \end{array} \pmod{P^5}.$$

To illustrate the difference between types 6 and 7 it will be better to take groups of higher order. E. g., put $m = 3$, $k = 8$, and $P^8 = (16, 16 + 16\sqrt{3})$; then G is of type 6 and order 128 and is generated by $\alpha = \sqrt{3}$ of period 8, $\beta = 1 + 2\sqrt{3}$ of period 8, and $c = -1$ of period 2. The subgroup $\{\alpha, c\}$ of order 16 may be written out as follows:

$$\begin{array}{llll} 1 = 1, & \alpha = \sqrt{3}, & c \equiv 15, & \alpha c \equiv 15\sqrt{3}, \\ \alpha^2 \equiv 3, & \alpha^3 \equiv 3\sqrt{3}, & \alpha^2 c \equiv 13, & \alpha^3 c \equiv 13\sqrt{3}, \\ \alpha^4 \equiv 9, & \alpha^5 \equiv 9\sqrt{3}, & \alpha^4 c \equiv 7, & \alpha^5 c \equiv 7\sqrt{3}, \\ \alpha^6 \equiv 11, & \alpha^7 \equiv 11\sqrt{3}, & \alpha^6 c \equiv 5, & \alpha^7 c \equiv 5\sqrt{3}, \end{array} \pmod{P^8}.$$

On the other hand, if $m = -1$, $k = 8$, and $P^8 = (16, 16 + 16i)$, then G is of the same order 128, but of type 7, and is generated by $\alpha = i$ of period 4, $\beta = 1 + 2i$ of period 8, and $c = 5$ of period 4. The subgroup $\{\alpha, c\}$ of

order 16 may be written out as follows:

$$\begin{array}{llll} 1 = 1, & \alpha = i, & \alpha^2 \equiv 15, & \alpha^3 \equiv 15i, \\ c = 5, & \alpha c \equiv 5i, & \alpha^2 c \equiv 11, & \alpha^3 c \equiv 11i, \\ c^2 \equiv 9, & \alpha c^2 \equiv 9i, & \alpha^2 c^2 \equiv 7, & \alpha^3 c^2 \equiv 7i, \\ c^3 \equiv 13, & \alpha c^3 \equiv 13i, & \alpha^2 c^3 \equiv 3, & \alpha^3 c^3 \equiv 3i, \end{array} \pmod{P^3}.$$

Case III. *P a prime ideal of the second grade.*

29. In this case $P = (p) = (p, p\omega)$, that is, the prime ideal is a rational principal ideal, and its norm is p^2. Further, $P^k = (p^k) = (p^k, p^k\omega)$. Since ω and p are relatively prime, every quadratic integer $u + v\omega$ is prime to P^k unless u and v are both divisible by p. The number of residues in G is therefore $p^{2k-2}(p^2 - 1)$. If $k > 1$, the residues $1 + p(u + v\omega)$ obviously form a Sylow subgroup H of order p^{2k-2} comprising all the elements whose periods are powers of p.

Since d, the discriminant of the field, is a non-square modulo p, the congruence $x^2 - d \equiv 0 \pmod{p}$ is irreducible and defines a *Galois field* of order p^2 whose elements may therefore be concretely represented by a complete system of residues $u + v\omega$ with respect to the modulus $(p, p\omega)$. But these residues, excepting 0, are precisely the elements of the group G in the case $k = 1$; hence the latter is a cyclic group of order $p^2 - 1$.

30. *Therefore any complete system of residues with respect to a prime ideal P of the second grade constitutes a Galois field F of order* p^2. It goes without saying that a complete system of residues with respect to a prime ideal of the first grade may also be looked upon as a Galois field of order p, and that with respect to a composite ideal the residues do not form a field at all.

Let δ_1 be a primitive root of the Galois field F, or in other words a generator of the group G, mod P. With respect to the modulus P^k, $\delta_1^{p^2-1}$ will be of period a power of p, say p^i, and $\delta = \delta_1^{p^i}$ will be of period $p^2 - 1$. We shall therefore choose δ to be a generator of the group G, mod P^k. In order to generate the entire group, it will only be necessary in addition to find generators of the subgroup H. At this point two cases must be distinguished, the case $p > 2$ in which G is of type 8, and the case $p = 2$ in which G is of type 9.

Type 8. $p > 2$.

31. In this case H is evidently generated by the two independent elements $1 + p$, and $1 + p\omega$, both of period p^{k-1}; hence *the entire group G, if* $k > 1$, *has the three independent generators* δ, $1 + p$, *and* $1 + p\omega$, *whose periods are* $p^2 - 1$, p^{k-1}, *and* p^{k-1}, *respectively.*

The elements having rational residues form a subgroup of order $p^{k-1}(p-1)$ generated by δ^{p+1} of period $p-1$, and $1+p$ of period p^{k-1}. It is easy to see that the system of residues of the form $1+p^iu+p^jv\omega$, where i and j are fixed exponents such that $1 \leqq i \leqq k$ and $1 \leqq j \leqq k$, constitute a group (a subgroup of H of order p^{2k-i-j}), if and only if $i \leqq 2j$. An element α is of period p^i $(i<k)$, if and only if it is expressible in the form

$$\alpha \equiv 1+p^{k-i}(u+v\omega) \pmod{P^k},$$

where u and v are not both divisible by p.

Type 9. $p=2$.

32. In this case $m \equiv 5 \pmod 8$, $\omega = \frac{1}{2}(1+\sqrt{m})$, $P^k=(2^k, 2^k\omega)$, and $\Phi(P^k)=2^{2k-2}\cdot 3$. In the subgroup H whose elements are of the form $1+2(u+v\omega)$, the rational elements $1, 3, \cdots, 2^k-1$, which are now the only rational elements in the entire group G, are generated as usual by -1 and 5; but the irrational element $1+2\omega$ will no longer serve as the additional generator of H, as it is not independent of the rational elements. Now if $k>2$, $1+4\omega$ is plainly of period 2^{k-2} and independent of the rational elements, so that -1, 5, and $1+4\omega$ generate just half of H.

Since m, being a non-square (mod 8), is of period 2^{k-2} $(k>2)$, $\sqrt{m}=-1+2\omega$ is an irrational element of period 2^{k-1} independent of -1 and $1+4\omega$. So H is generated by $\sqrt{m}$, -1, and $1+4\omega$, if $k>2$, and evidently by $\sqrt{m}$ and -1, if $k=2$.

Summing up, we may say that *if P is a prime ideal of the second grade whose norm is* 4, *then in the group of reduced residues, mod P^k, the element δ defined in* § 30 *is an independent generator of period* 3; *if $k>1$ the elements $\sqrt{m}$ and -1 are independent generators of periods 2^{k-1} and* 2, *respectively; and if $k>2$, $1+4\omega$ is an independent generator of period 2^{k-2}.*

33. It may easily be proved that every residue of the form $1+2^{k-i}(u+v\omega)$, where $i<k$ and where u and v are not both even, is of period 2^i, except in the case where $i=k-1$, u is odd, and v is even.

While the generator δ must in general be determined separately for each value of k, there is one simple case in which the same determination of δ will hold for all values of k. Namely, if $m=-3$, $-\omega=-\frac{1}{2}(1+\sqrt{-3})$ is of period 3 irrespective of k, and can be taken to be the generator δ.

34. *Examples.* To illustrate type 8, we take $m=-1$, $p=3$, and therefore $\omega=\sqrt{-1}$, $P^k=(3^k, 3^k\omega)$. When $k=1$, G is cyclic, of order 8, and generated by $\delta=1+\omega$. When $k=2$, $\delta_1=1+\omega$ is of period 24, $\delta=\delta_1^3\equiv -2+2\omega$ is of period 8, and G, of order 72, is generated by δ, 4, and $1+3\omega$, the last two being of period 3; the rational residues are generated

by $\delta^4 \equiv -1$ and 4; the powers of δ are

$$\begin{aligned} 1 &= 1, & \delta &= -2+2\omega, \\ \delta^2 &\equiv \omega, & \delta^3 &\equiv -2-2\omega, \\ \delta^4 &\equiv -1, & \delta^5 &\equiv 2-2\omega, \\ \delta^6 &\equiv -\omega, & \delta^7 &\equiv 2+2\omega, \end{aligned} \pmod{P^2}.$$

To illustrate type 9, we take $m=-3$ and therefore $\omega = \frac{1}{2}(1+\sqrt{-3})$ and $P^k = (2^k, 2^k\omega)$. When $k=2$, G is of order 12 and is generated by $\delta = -\omega$, $\epsilon = \sqrt{-3}$, and $b=-1$, of periods 3, 2, and 2, respectively; written out in full, it is

$$\begin{aligned} &1=1, && \epsilon = -1+2\omega, && b=-1, && \epsilon b \equiv 1+2\omega, \\ &\delta = -\omega, && \delta\epsilon \equiv 2-\omega, && \delta b \equiv \omega, && \delta\epsilon b \equiv 2+\omega, \\ &\delta^2 \equiv -1+\omega, && \delta^2\epsilon \equiv -1-\omega, && \delta^2 b \equiv 1-\omega, && \delta^2\epsilon b \equiv 1+\omega, \end{aligned} \pmod{P^2}.$$

When $k=3$, G is of order 48 and is generated by $\delta = -\omega$ of period 3, $\sqrt{-3}$ of period 4, -1 of period 2, and $1+4\omega$ of period 2.

Table of Moduli, Elements, and Generators.

35. For convenience of reference we shall now exhibit in tabular form the principal results already obtained, namely those relating to the nine types of groups, mod P^k, and in addition, for the sake of completeness, those relating to two other types, the tenth and eleventh, to be treated in §§ 46–51.

Modulus $A = P^k$.

I. P unambiguous, of the first grade; $PP' = (p)$, $(d/p) = 1$.

Type 1. $p>2$, $m \equiv 1, 2,$ or $3 \pmod 4$. $P=(p, a_1+\omega)$, $P^k=(p^k, a_k+\omega)$, $\Phi(P^k) = p^{k-1}(p-1)$. Elements $u = 1, 2, \cdots, p^k-1$; $u \not\equiv 0 \pmod p$. Generators f, defined in § 5, and $1+p$; periods $p-1$ and p^{k-1}.

Type 2. $p=2$, $m \equiv 1 \pmod 8$. $P=(2, \omega)$, $P^k=(2^k, a_k+\omega)$, $\Phi(P^k)=2^{k-1}$. Elements $u = 1, 3, \cdots, 2^k-1$. Generators -1 and 5; periods 2 and 2^{k-2}.

II. P ambiguous, of the first grade; $P^2 = (p)$, $(d/p) = 0$.

II$_a$. $p>2$, $m \equiv 1, 2,$ or $3 \pmod 4$.

Type 3. Either $p>3$ and $m \equiv 0 \pmod p$, or $p=3$ and $m \equiv 3 \pmod 9$. $P = (p, \omega_1)$, $P^k = (p^r, p^s\omega_1)$, $\Phi(P^k) = p^{k-1}(p-1)$. Elements $\alpha = u + v\sqrt{m}$; $u = 1, 2, \cdots, p^r-1$; $u \not\equiv 0 \pmod p$; $v = 0, 1, \cdots, p^s-1$. Generators g, defined in § 11, $1+\sqrt{m}$, and $1+p$; periods $p-1$, p^s, and $1+p^{r-1}$.

Type 4. $p=3$ and $m\equiv -3 \pmod 9$. $P=(3,\ \omega_1)$, $P^k=(3^r,\ 3^s\omega_1)$, $\Phi(P^k)=3^{k-1}\cdot 2$. Elements $\alpha=u+v\sqrt{m}$; $u=1, 2, \cdots, 3^r-1$; $u \not\equiv 0 \pmod 3$; $v=0, 1, \cdots, 3^s-1$. Generators -1, λ, defined in §17, 4, and $1+3\sqrt{m}$; periods 2, 3, 3^{r-1}, and 3^{s-1}, if $k>1$; 2, 1, 1, and 1, if $k=1$.

II$_b$. $p=2$.

Type 5. $m\equiv 2 \pmod 4$. $P=(2, \sqrt{m})$, $P^k=(2^r, 2^s\sqrt{m})$, $\Phi(P^k)=2^{k-1}$. Elements $\alpha=u+v\sqrt{m}$; $u=1, 3, \cdots, 2^r-1$; $v=0, 1, \cdots, 2^s-1$. Generators $1+\sqrt{m}$, -1, and 5; periods 2^s, 2, and 2^{r-2}, if $k>2$; 2, 1, and 1, if $k=2$.

Type 6. $m\equiv 3 \pmod 8$. $P=(2, 1+\sqrt{m})$, $P^k=(2^r, 2^s+2^s\sqrt{m})$, $\Phi(P^k)=2^{k-1}$. Elements $\alpha=u+v\sqrt{m}$; $u=0, 1, \cdots, 2^r-1$; $v=0, 1, \cdots, 2^s-1$; $u+v\equiv 1 \pmod 2$. Generators $\sqrt{m}$, $1+2\sqrt{m}$, and -1; periods 2^{r-1}, 2^{s-1}, and 2, if $k>4$; 2^r, 2^{s-1}, and 1, if $k=2$, 3, or 4.

Type 7. $m\equiv 7 \pmod 8$. $P=(2, 1+\sqrt{m})$, $P^k=(2^r, 2^s+2^s\sqrt{m})$, $\Phi(P^k)=2^{k-1}$. Elements $\alpha=u+v\sqrt{m}$, as in type 6. Generators μ, defined in § 26, $1+2\sqrt{m}$, and 5; periods 4, 2^{s-1}, and 2^{r-2}, if $k>2$; 2, 1, and 1, if $k=2$.

III. P of the second grade; $P=(p)$, $(d/p)=-1$.

Type 8. $p>2$, $m\equiv 1$, 2, or 3 $\pmod 4$. $P=(p, p\omega)$, $P^k=(p^k, p^k\omega)$, $\Phi(P^k)=p^{2k-2}(p^2-1)$. Elements $\alpha=u+v\omega$; $u, v=0, 1, \cdots, p^k-1$; u and v not both $\equiv 0 \pmod p$. Generators δ, defined in § 30, $1+p$, and $1+p\omega$; periods p^2-1, p^{k-1}, and p^{k-1}.

Type 9. $p=2$, $m\equiv 5 \pmod 8$. $P=(2, 2\omega)$, $P^k=(2^k, 2^k\omega)$, $\Phi(P^k)=2^{2k-2}\cdot 3$. Elements $\alpha=u+v\omega$; $u, v=0, 1, \cdots, 2^k-1$; u and v not both even. Generators δ, defined in § 30, $\sqrt{m}=-1+2\omega$, -1, and $1+4\omega$; periods 3, 2^{k-1}, 2, and 2^{k-2}, if $k>1$; 3, 1, 1, and 1, if $k=1$.

Modulus $A=P^kP'^k$, $P\neq P'$.

P unambiguous, of the first grade; $PP'=(p)$, $(d/p)=1$.

Type 10. $p>2$, $m\equiv 1$, 2, or 3 $\pmod 4$. $P=(p, a+\sqrt{m})$, if $m\equiv 2$ or 3 $\pmod 4$; $P=(p, \frac{1}{2}a+\frac{1}{2}\sqrt{m})$, if $m\equiv 1 \pmod 4$. $P^kP'^k=(p^k)$, $\Phi(A)=p^{2k-2}(p-1)^2$. Elements $\alpha=u+v\sqrt{m}$; $u, v=0, 1, \cdots, p^k-1$; $u \not\equiv \pm av \pmod p$. Generators h, defined in § 48, ζ, defined in § 48, $1+p$, and $1+p\sqrt{m}$; periods $p-1$, $p-1$, p^{k-1}, and p^{k-1}.

Type 11. $p=2$, $m\equiv 1 \pmod 8$. $P=(2, \frac{1}{2}+\frac{1}{2}\sqrt{m})$, $P^kP'^k=(2^k)$, $\Phi(A)=2^{2k-2}$. Elements $\alpha=u+v\sqrt{m}$; $u, v=0, 1, \cdots, 2^k-1$; $u+v\equiv 1 \pmod 2$. Generators -1, θ, defined in § 50, 5, and $1+2\sqrt{m}$; periods 2, 2, 2^{k-2}, and 2^{k-2}.

Modulus any Composite Ideal.

36. Having found the structure of the group G in every case in which the modulus is a power of a prime ideal, we now have sufficient material at hand for constructing the group in the general case in which the modulus is any composite ideal whatever. This is easily accomplished in a similar manner to that used in the rational number-field.*

Let $A = P_1^{k_1} \cdots P_n^{k_n}$ be any ideal whose distinct prime factors are $P_1, \cdots, P_n$. Then the order of the group G formed by a reduced system of residues with respect to the modulus A is $\Phi(A) = \Phi(P_1^{k_1}) \cdots \Phi(P_n^{k_n})$ and the group itself is the direct product of n subgroups $G_i\,(i = 1, \cdots, n)$, each of which is simply isomorphic with a corresponding group $\bar{G}_i$, whose elements form a reduced system of residues, mod $P_i^{k_i}$. Every quadratic integer α, prime to A, uniquely determines n residues $\alpha_1, \cdots, \alpha_n$, belonging respectively to $\bar{G}_1, \cdots \bar{G}_n$, which satisfy the congruences

$$\alpha \equiv \alpha_1 \pmod{P_1^{k_1}}, \quad \cdots, \quad \alpha \equiv \alpha_n \pmod{P_n^{k_n}}. \tag{10}$$

When every residue α_i ranges over the elements of its group $\bar{G}_i$, α, determined uniquely by congruences (10), will range once and only once over all the elements of the group G. When every $\alpha_j = 1\ (j \neq i)$, and α_i ranges over the elements of its group $\bar{G}_i$, α will range over the elements of the subgroup G_i $(i = 1, \cdots, n)$.

37. Let $\bar{\lambda}_i$, $\bar{\mu}_i$, etc., be a *complete set of independent generators* of the group $\bar{G}_i\,(i = 1, \cdots, n)$, and let their periods be l_i', m_i', etc., respectively. Then every element α_i of $\bar{G}_i$ is expressible in the form $\alpha_i \equiv \bar{\lambda}_i^{l_i} \bar{\mu}_i^{m_i} \cdots \pmod{P_i^{k_i}}$, where the exponents l_i, m_i, etc., are residues with respect to the corresponding periods, l_i', m_i', etc. The congruences (10) now become

$$\begin{aligned} \alpha &\equiv \bar{\lambda}_1^{l_1} \bar{\mu}_1^{m_1} \cdots \pmod{P_1^{k_1}}, \\ &\cdots\cdots\cdots \\ \alpha &\equiv \bar{\lambda}_n^{l_n} \bar{\mu}_n^{m_n} \cdots \pmod{P_n^{k_n}}. \end{aligned}$$

Corresponding to the generators $\bar{\lambda}_i$, $\bar{\mu}_i$, etc., of the groups $\bar{G}_i\,(i = 1, \cdots, n)$, we now define generators λ_i, μ_i, etc., of the subgroups $G_i\,(i = 1, \cdots, n)$, and therefore of the entire group G, by means of the congruences

$$\begin{aligned} \lambda_i &\equiv \bar{\lambda}_i \pmod{P_i^{k_i}} \\ &\equiv 1 \pmod{P_j^{k_j}} \quad (j = 1, \cdots, n;\ j \neq i), \\ \mu_i &\equiv \bar{\mu}_i \pmod{P_i^{k_i}} \\ &\equiv 1 \pmod{P_j^{k_j}} \quad (j = 1, \cdots, n;\ j \neq i), \\ &\cdots\cdots\cdots \end{aligned} \qquad (i = 1, \cdots, n).$$

* Cf. Weber, *Algebra*, vol. 2, 2nd edition, pp. 60–61.

The generators $\lambda_1, \mu_1, \cdots, \lambda_n, \mu_n$, etc., so defined, are of periods $l_1', m_1', \cdots, l_n', m_n'$, etc., and form a complete set of independent generators of G, in terms of which every element α of the group can be written as follows

$$\alpha \equiv (\lambda_1^{l_1}\mu_1^{m_1}\cdots), \cdots, \alpha \equiv (\lambda_n^{l_n}\mu_n^{m_n}\cdots), \mod A.$$

The exponents $l_1, m_1, \cdots, l_n, m_n$, will be called the *indices* of the quadratic integer α with respect to the modulus A.

This completes the solution of the general problem of determining the group of classes of congruent quadratic integers with respect to any composite modulus. But there are certain special cases that merit separate treatment.

Cyclic Groups.

38. If the group G, whose modulus is the ideal A, can be generated by a single element η and is therefore *cyclic*, η is said to be a *primitive root* of A. The number of primitive roots of A, if there is one such, is $\phi[\Phi(A)]$, where ϕ is Euler's ϕ-function. If G is not cyclic, A does not possess primitive roots.

The object of this part of the paper, §§ 38–45 inclusive, is to enumerate all the cases in which G is cyclic and to give at least one primitive root in every case. G will be cyclic, (1) if there is only one independent generator, and (2) if the periods of each pair of independent generators are relatively prime. In examining for cyclic groups, therefore, it will be well to note (1) that p and $p-1$ are relatively prime, (2) if p is an odd prime, $p-1$ and p^2-1 are even, and (3) if $p>3$, p^2-1 is divisible by 3.

Case I. *Modulus a Power of a Prime Ideal;* $A = P^k$.

39. (a) Ideal P unambiguous, of the first grade.

(a_1) $p>2$, $m \equiv 1, 2,$ or $3 \pmod 4$. Modulus $P^k=(p^k, a_k+\omega)$. In this case G is of type 1 and is cyclic for all values of k. We choose as primitive root of A the rational integer $f(1+p)$, of period $p^{k-1}(p-1)$, where f is defined in § 5.

(a_2) $p=2$, $m \equiv 1 \pmod 8$. Here G is of type 2 and is cyclic, only if $k=2$ or 1.

Modulus $P^2=(4, 2a+\omega)$. Primitive root -1, of period 2.

Modulus $P=(2, \omega)$. Primitive root 1, of period 1. This case would be trivial except for its application to Case II.

(b) Ideal P ambiguous, of the first grade.

(b_1) $p>2$, $m \equiv 1, 2,$ or $3 \pmod 4$, $m \equiv 0 \pmod p$. G is of type 3 or 4, and can only be cyclic when $k=2$ or 1.

Modulus $P^2=(p, p\omega)$. Primitive root $g(1+\sqrt{m})$ (§ 11), of period $p(p-1)$.

Modulus $P=(p, \omega_1)$. Primitive root g, of period $p-1$.

(b_2) $p=2$, $m \equiv 2 \pmod 4$. G is of type 5, and is cyclic when $k=2$ or 1.

Modulus $P^2 = (2, 2\sqrt{m})$. Primitive root $1 + \sqrt{m}$, of period 2.

Modulus $P = (2, \sqrt{m})$. Primitive root 1, of period 1.

(b_3) $p = 2$, $m \equiv 3 \pmod 4$. G is of type 6 or 7 and is cyclic, when $k = 3$, 2, or 1.

Modulus $P^3 = (4, 2 + 2\sqrt{m})$. Primitive root $\sqrt{m}$, of period 4.

Modulus $P^2 = (2, 2 + 2\sqrt{m})$. Primitive root $\sqrt{m}$, of period 2.

Modulus $P = (2, 1 + \sqrt{m})$. Primitive root 1, of period 1.

(c) Ideal P of the second grade.

G is of type 8 or 9 and is cyclic only when $k = 1$.

Modulus $P = (p, p\omega)$. Primitive root δ, defined in § 30, of period $p^2 - 1$. In the two simplest cases in which $p = 3$ and $p = 2$, δ can be found easily by means of a formula. Namely, if $p = 3$, $1 + \sqrt{m}$ is always a primitive root of P, of period 8, and if $m \equiv 1 \pmod 4$, $\omega = \frac{1}{2}(1 + \sqrt{m})$ is also a primitive root; again, if $p = 2$ and therefore $m \equiv 5 \pmod 8$, $\omega = \frac{1}{2}(1 + \sqrt{m})$ is a primitive root of P, of period 3.

40. Summarizing these results, we observe first that G is always cyclic when the modulus is the first power of P whether P is of the first or second grade. This agrees with the well-known fact that every prime ideal possesses primitive roots. Next we observe that *when P is a prime ideal of the first grade, whether ambiguous or not, P^2 as well as P possesses primitive roots; and when P is an ambiguous prime whose norm is 2, and $m \equiv 3 \pmod 4$, then P, P^2, and P^3 all have primitive roots. Finally, the only case in which P^k has primitive roots for all values of k is that in which P is an unambiguous prime of the first grade whose norm p exceeds 2.*

Case II. *Cyclic Groups with Modulus Divisible by Two Distinct Prime Ideals;* $A = P^k Q^l$.

41. To obtain the moduli that come under this case, we multiply together any two (more than two are evidently impossible) prime-power moduli coming under Case I, which occur for the same value of m and which are so related that the periods of their primitive roots are relatively prime. If P^k and Q^l are two such moduli, and if ρ, of period r, and σ, of period s, are a pair of their respective primitive roots, then a primitive root η of the modulus $A = P^k Q^l$ is defined by the congruences

$$\eta \equiv \rho \pmod{P^k}, \qquad \eta \equiv \sigma \pmod{Q^l}.$$

For, since r and s are relatively prime, η is of period rs.

42. Examining the moduli of case 1, we find that they fall into three classes: first, those whose primitive roots are of period 3, namely those for which P is of the second grade, $p = 2$, $k = 1$, and therefore $m \equiv 5 \pmod 8$; second, those

whose primitive roots are of period 1, namely those for which P is of the first grade, $p = 2$, $k = 1$, and therefore $m \equiv 1, 2, 3, 6$, or $7 \pmod 8$; and third, those whose primitive roots are of even period, namely all the remainder of the moduli.

Now it is clear that for a given value of m there exists just one modulus (or at most two, in a certain exceptional case) of either the first or second class, and an infinite number of moduli of the third class. But the product of two moduli of the third class will not have primitive roots. Therefore, to obtain a modulus having primitive roots we are restricted (with one exception) to multiplying a modulus of the third class by the particular modulus of the first or second class that happens to exist for the given value of m.

Subcase II_a. $m \equiv 5 \pmod 8$.

43. In this case there exists a modulus of the first class, namely a prime ideal $Q = (2, 2\omega)$ of the second grade having a primitive root $\omega = \frac{1}{2}(1 + \sqrt{m})$ of period 3. We proceed to construct moduli of the form $P^k Q$ by selecting from the third class an ideal P^k such that p > 2 and the period of its primitive roots is not divisible by 3. If ρ is a primitive root of P^k, then the congruences

$$\eta \equiv \rho \pmod{P^k}, \qquad \eta \equiv \omega \pmod{Q}$$

determine a primitive root η of $P^k Q$.

(a) Ideal P unambiguous, of the first grade, $(m/p) = 1$.

(a_1) $p > 3$, $p \equiv 2 \pmod 3$, $P^k = (p^k, a_k + \omega)$.

Modulus $P^k Q = (2p^k, 2a_k + 2\omega)$. Primitive root $\eta = f(1 + p) + ep^k + a_k + \omega$ of period $3p^{k-1}(p - 1)$, where f is defined in § 5, and $e = 0$ or 1 according as a_k is even or odd.

(a_2) $p = 3$, $m \equiv 13 \pmod{24}$, $P = (3, \omega)$.

Modulus $PQ = (6, 2\omega)$. Primitive root $\eta = 2 + \omega$ of period 6.

(b) Ideal P ambiguous, of the first grade, $m \equiv 0 \pmod p$.

(b_1) $p > 3$, $p \equiv 2 \pmod 3$, $P = (p, \frac{1}{2}p - \frac{1}{2} + \omega)$, $P^2 = (p, p\omega)$.

Modulus $P^2 Q = (2p, 2p\omega)$. Primitive root $\eta = (2g + p)\omega$ (see § 11) of period $3p(p - 1)$.

Modulus $PQ = (2p, p - 1 + 2\omega)$. Primitive root $\eta = g + ep + \frac{1}{2}(p - 1) + \omega$ of period $3(p - 1)$, where $e = 0$ or 1 according as $g + \frac{1}{2}(p - 1)$ is even or odd.

(b_2) $p = 3$, $m \equiv -3 \pmod{24}$, $P = (3, 1 + \omega)$.

Modulus $PQ = (6, 2 + 2\omega)$. Primitive root $\eta = \omega$, of period 6.

(c) Ideal P of the second grade.

$p = 3$, $m \equiv 5 \pmod{24}$, $P = (3, 3\omega)$.

Modulus $PQ = (6, 6\omega)$. Primitive root ω of period 24.

Subcase II_b. $m \equiv 1, 2, 3, 6$, or $7 \pmod 8$.

44. In this case there exists a modulus of the second class, namely a prime ideal $Q = (2, b + \omega)$ of the first grade, having a primitive root 1 of period 1. Any modulus of the form $P^k Q$ for which P^k belongs to the third class and $p > 2$ will obviously have a primitive root η whose period is equal to that of the primitive roots of P^k. If ρ is one of the latter, then η can be defined by the congruences

$$\eta \equiv \rho \pmod{P^k}, \qquad \eta \equiv 1 \pmod{Q}.$$

(a) Ideal P unambiguous, of the first grade, $p > 2$, $P^k = (p^k, a_k + \omega)$.

Modulus $P^k Q = (2p^k, b_k + \omega)$. Primitive root $\eta = f(1 + p) + p^k$ (see § 5), of period $p^{k-1}(p - 1)$.

In the two remaining cases, in which P is ambiguous or of the second grade, it is evident that η will be equal to ρ or $\rho + p$; therefore in every case $\eta = \rho + ep$, where $e = 0$ or 1.

(b) Ideal P ambiguous, $p > 2$, $m \equiv 0 \pmod p$, $P = (p, a + \omega)$, $P^2 = (p, p\omega)$.

Modulus $P^2 Q = (2p, bp + p\omega)$. Primitive root $\eta = g(1 + \sqrt{m}) + ep$ (see § 11), of period $p(p - 1)$.

Modulus $PQ = (2p, c + \omega)$. Primitive root $\eta = g + ep$, of period $p - 1$.

(c) Ideal P of the second grade, $p > 2$, $P = (p, p\omega)$.

Modulus $PQ = (2p, bp + p\omega)$. Primitive root $\eta = \delta + ep$ (see § 30), of period $p^2 - 1$.

Subcase II_c.

45. This is the exceptional case mentioned in § 42, in which the modulus is divisible by two distinct conjugate primes P and P'. P is therefore an unambiguous prime of the first grade, $p = 2$, and $m \equiv 1 \pmod 8$. $P = (2, \omega)$ and $P' = (2, \omega')$.

Modulus $P^2 P' = (4, 2\omega)$. Primitive root -1, of period 2.

Modulus $PP' = (2, 2\omega)$. Primitive root 1, of period 1.

We have now exhausted all the possible cases of cyclic groups.

Modulus $(p^k) = P^k P'^k$.

46. A rational prime p, considered as a principal ideal (p), may be either a prime ideal of the second grade, or the square of an ambiguous prime of the first grade, or finally the product of an unambiguous prime of the first grade and its conjugate. The corresponding group of residues with respect to the modulus (p^k) has been considered for the first two cases in §§ 29–34 and 7–28, respectively. For the last case, in which the modulus contains two distinct conjugate primes, the general procedure of §§ 36, 37 is applicable, but a special direct method will now be developed, giving simpler results.

Let P be any unambiguous prime ideal of the first grade and P' its conjugate; then $(d/p) = 1$, and $PP' = (p) = (p, p\omega)$. Let

$$A = P^k P'^k = (p^k) = (p^k, p^k\omega),$$

and let G be the group of residues prime to the modulus A. G will be of two distinct types, depending on the value of p. Continuing the former numbering, we shall call them types 10 and 11. The results to be obtained have already been tabulated in § 35.

Type 10. $p > 2$.

47. In this case m may be $\equiv 1, 2$, or $3 \pmod 4$, and $P = (p, a_1 + \omega)$, $P' = (p, a_1 + \omega')$. If $m \equiv 2$ or $3 \pmod 4$, $\omega = \sqrt{m}$, $\omega' = -\sqrt{m}$, and, putting $a_1 = a$, we have

$$P = (p, a + \sqrt{m}), \qquad P' = (p, a - \sqrt{m}). \tag{11}$$

If $m \equiv 1 \pmod 4$,

$$a_1 + \omega = \frac{2a_1 + 1 + \sqrt{m}}{2}, \qquad a_1 + \omega' = \frac{2a_1 + 1 - \sqrt{m}}{2};$$

putting $2a_1 + 1 = a$, we have

$$P = \left(p, \frac{a + \sqrt{m}}{2}\right), \qquad P' = \left(p, \frac{a - \sqrt{m}}{2}\right). \tag{12}$$

Instead of writing the elements of G in the form $u + v\omega$, it will be allowable and convenient, exactly as in the similar case of § 10, to write them in the form $u + v\sqrt{m}$, where u and v are rational integers, residues with respect to the modulus p^k.

Since every element $u + v\sqrt{m}$ must be prime to P and also to P', therefore in view of (11) and (12) u and v must satisfy the condition $u \not\equiv \pm av \pmod p$. This gives $p^{2k-2}(p-1)^2$ as the order of the group, in accord with the value of $\Phi(A)$ and also with the fact (see § 36) that G is the direct product of two cyclic subgroups of order $p^{k-1}(p-1)$, which are simply isomorphic with groups whose moduli are P^k and P'^k, respectively.

48. As generators we could choose two independent elements of period $p^{k-1}(p-1)$, but we shall rather, in accordance with our former rule, choose four independent generators, two of period $p-1$ and, if $k > 1$, two others of period p^{k-1}. It is obvious that the two latter can be chosen to be $1 + p$ and $1 + p\sqrt{m}$, and that they generate a subgroup H, of order p^{2k-2}, comprising the residues $1 + pu + pv\sqrt{m}$, where u and v range from 0 to $p^{k-1} - 1$.

Let h be a rational residue of period $p-1$, like the generator f in type 1 (§ 5), and let ζ be a residue, necessarily irrational, satisfying the congruences

$$\zeta \equiv h \pmod{P^k}, \qquad \zeta \equiv 1 \pmod{P'^k}. \tag{13}$$

Then ζ is also of period $p-1$ and is evidently independent of h (cf. § 37).

Hence G is generated by the following four elements: h and ζ, both of period $p-1$, and if $k>1$, $1+p$ and $1+p\sqrt{m}$, both of period p^{k-1}. The rational residues are generated by $1+p$ and h; if we add to the rational residues the rational multiples of $\sqrt{m}$, we obtain a subgroup of order $2p^{k-1}(p-1)$, which is generated by $1+p$, h, and $\zeta^{\frac{1}{2}(p-1)}$. Every residue of period p^i $(0<i<k)$ is of the form $1+p^{k-i}(u+v\sqrt{m})$, where u and v are not both divisible by p.

49. *Example.* Let $m=-1$, $p=5$, and $k=1$; then $P=(5, 2+i)$, $P'=(5, 2-i)$, $A=PP'=(5, 5i)$, and $\Phi(A)=16$. The elements of G are $u+vi$, where $u, v=0, \pm 1, \pm 2$, and $u \not\equiv \pm 2v \pmod 5$. Its generators are $h=2$ and $\zeta=-1+i$, defined by (13), both of period 4. In terms of h and ζ, G may be written out, as follows:

$$\begin{array}{llll} 1=1, & \zeta=-1+i, & \zeta^2\equiv -2i, & \zeta^3\equiv 2+2i, \\ h=2, & h\zeta\equiv -2+2i, & h\zeta^2\equiv i, & h\zeta^3\equiv -1-i, \\ h^2\equiv -1, & h^2\zeta\equiv 1-i, & h^2\zeta^2\equiv 2i, & h^2\zeta^3\equiv -2-2i, \\ h^3\equiv -2, & h^3\zeta\equiv 2-2i, & h^3\zeta^2\equiv -i, & h^3\zeta^3\equiv 1+i, \end{array} \pmod{PP'}.$$

If we had used the general method of § 37, the generators would have been $-1+i$ and $-1-i$.

Type 11. $p=2$.

50. In this case $m\equiv 1 \pmod 8$, $P=(2, \omega)=(2, \frac{1}{2}+\frac{1}{2}\sqrt{m})$, and $P'=(2, \omega')=(2, \frac{1}{2}-\frac{1}{2}\sqrt{m})$; also $A=(2^k, 2^k\omega)=(2^k, 2^{k-1}+2^{k-1}\sqrt{m})$. Since the residues $u+v\sqrt{m}$ are subject to the condition $u \not\equiv v \pmod 2$, it follows that either u is odd and v even, or u is even and v odd. Except for this restriction u ranges from 0 to 2^k-1 and v ranges from 0 to $2^{k-1}-1$. G is therefore of order 2^{2k-2}, which agrees with the value of $\Phi(A)$. If $k>1$, the residues for which u is odd and v even, namely those of the form $1+2u+2v\sqrt{m}$, obviously constitute a subgroup H of index 2 and order 2^{2k-3}. H is generated by the rational residues and the irrational residue $1+2\sqrt{m}=-1+4\omega$, of period 2^{k-2}; and the former are generated, as usual, by -1 and 5, of periods 2 and 2^{k-2}, respectively. Finally, G is generated by H and a single element θ of period 2 of the form $2u+(2v+1)\sqrt{m}$; if $k>2$, *θ can be chosen in eight different ways*, the simplest of which is the following:

$$\theta = x\sqrt{m}, \tag{14}$$

where x is a root of the congruence

$$mx^2 \equiv 1 \pmod{2^k}. \tag{15}$$

This congruence can always be solved for x, since m, being congruent to

1 (mod 8), is necessarily a square, mod 2^k ($k > 2$); if $k = 2$, the same is true. Therefore, if $k > 1$, θ is a generator of period 2 of the required form.

Summing up, we may say that *the group G of reduced residues with respect to the modulus $(2^k) = P^k P'^k$ is of order* 1, *if* $k = 1$ (cf. § 45, cyclic groups); *if* $k > 1$, *it has the two independent generators* -1, *of period* 2, *and* θ *defined by* (14) *and* (15), *also of period* 2; *if* $k > 2$, *it has the two additional independent generators* 5 *and* $1+2\sqrt{m}$, *both of period* 2^{k-2}.

The residues $1 + 4u + 2v\sqrt{m}$ evidently form a subgroup of order 2^{2k-4} generated by 5 and $1 + 2\sqrt{m}$.

51. *Example.* If $m = -7$ and $k = 3$, then $P = (2, \frac{1}{2} + \frac{1}{2}\sqrt{-7})$ and $A = P^3 P'^3 = (8, 4 + 4\sqrt{-7})$. G is of order 16 and is generated by the four independent elements $a = -1$, $b = 5$, $\theta = \sqrt{-7}$, and $\lambda = 1 + 2\sqrt{-7}$, all of period 2. Thus:

$$\begin{array}{llll} 1\equiv 1, & \theta\equiv\sqrt{-7}, & \lambda\equiv 1+2\sqrt{-7}, & \theta\lambda\equiv 2+\sqrt{-7}, \\ a\equiv 7, & a\theta\equiv 4+3\sqrt{-7}, & a\lambda\equiv 3+2\sqrt{-7}, & a\theta\lambda\equiv 2+3\sqrt{-7}, \\ b\equiv 5, & b\theta\equiv 4+\sqrt{-7}, & b\lambda\equiv 5+2\sqrt{-7}, & b\theta\lambda\equiv 6+\sqrt{-7}, \\ ab\equiv 3, & ab\theta\equiv 3\sqrt{-7}, & ab\lambda\equiv 7+2\sqrt{-7}, & ab\theta\lambda\equiv 6+3\sqrt{-7}, \end{array} \pmod{P^3 P'^3}.$$

Considerations of Rationality.

52. We shall now take up certain considerations of rationality that naturally give rise to two special kinds of moduli. Let $A = (gl, hl + l\omega)$ be any ideal in the number-field $k(\sqrt{m})$; then $A = A_1 A_2$, where $A_1 = (l)$ and $A_2 = (g, h+\omega)$. Thus A_1 is a rational principal ideal and A_2 will be called a *completely irrational ideal*, because it contains no rational factors. If the unit ideal (1) is regarded as belonging to both classes, then every ideal whatever can be factored in one and essentially only one way into two factors, one of which is a rational principal ideal and the other a completely irrational ideal. Evidently the norm of A is $n(A) = gl^2$, and $n(A_1) = l^2$, $n(A_2) = g$.

Let us examine the character of the prime factors of A_1 and of A_2. Since a completely irrational ideal A_2 cannot be divisible by a prime of the second grade, or by a power of an ambiguous prime of the first grade higher than the first, or finally by the conjugate of any of its unambiguous prime factors of the first grade, and since the converse evidently holds, therefore A_2 is characterized by being expressible in the form

$$A_2 = P_1^{\alpha_1} \cdots P_i^{\alpha_i} \cdot Q_1 \cdots Q_j, \tag{16}$$

where $P_1, \cdots, P_i$ are distinct unambiguous primes of the first grade, no two of which are conjugate, and $Q_1, \cdots, Q_j$ are distinct ambiguous primes.

53. On the other hand, if a rational principal ideal A_1 contains an unambig-

uous prime factor of the first grade, it must contain the same power of its conjugate, and if it contains an ambiguous prime factor Q, the highest power of Q which it contains must be an even power. Therefore A_1 is characterized by being expressible in the form

$$A_1 = (P_1 P_1')^{n_1} \cdots (P_i P_i')^{n_i} \cdot Q_1^{2e_1} \cdots Q_j^{2e_j} \cdot R_1^{r_1} \cdots R_k^{r_k}, \tag{17}$$

where P_1 and P_1' are conjugate unambiguous primes of the first grade, $Q_1, \cdots, Q_j$ are ambiguous primes, and $R_1, \cdots, R_k$ are primes of the second grade.

In other words, *an ideal A_1 is a rational principal ideal if, and only if, its unambiguous prime factors of the first grade enter in pairs of conjugates and its ambiguous prime factors are raised to even powers.*

We now turn to the consideration of the groups whose moduli are A_1 and A_2, respectively, and their special peculiarities. It is obvious that the group whose modulus is $A_1 A_2$ is not, in general, the direct product of two subgroups simply isomorphic with the groups whose moduli are A_1 and A_2, respectively.

Modulus a completely irrational ideal.

54. First, let G be the group of reduced residues with respect to a completely irrational modulus $A_2 = (g, h + \omega)$ whose prime factors are given by formula (16). Evidently $g = n(A_2) = p_1^{s_1} \cdots p_i^{s_i} \cdot q_1 \cdots q_j$, and the order of G is $\Phi(A_2) = \phi(g) = \phi[n(A_2)]$. Now since the coefficient of ω in the canonical form of A_2 is 1, the elements of G all have rational residues and G is simply isomorphic with the group of rational residues, mod g. Conversely, if A_2 is not completely irrational, the elements of G cannot all have rational residues. For by reference to the groups of types 8–11, and to those groups of types 3–7 in which $k > 1$, we see that they all contain elements whose residues are necessarily irrational. We have therefore proved the following theorem:

A necessary and sufficient condition that the elements of a group of quadratic integers with respect to an ideal modulus all have rational residues is that the modulus be a completely irrational ideal.

Modulus a rational principal ideal.

55. Again, consider the group G of reduced residues with respect to a rational principal ideal $A_1 = (l, l\omega)$ whose prime factors are given by formula (17). In this case the group is of peculiar interest because *its elements can be defined without using the language of ideals at all.* Thus the congruence $a + b\omega \equiv a' + b'\omega \pmod{A_1}$ is equivalent to the pair of rational congruences $a \equiv a', b \equiv b' \pmod{l}$, and the quadratic integer $a + b\omega$ will be prime to the ideal A_1 if, and only if, it is prime to the rational integer l. Therefore *G is really the group of classes of congruent quadratic integers with respect to a rational modulus l.*

56. In order to determine its structure when the field $k(\sqrt{m})$ and the rational integer l are given, we proceed as follows. First factor l and classify its prime factors into p's, q's, and r's in such a way that d, the discriminant of the field, is a square with respect to every p, a multiple of every q, and a non-square with respect to every r (if $p = 2$ or $r = 2$, d is a square or non-square, respectively, with respect to 8, instead of 2), or, briefly, so that $(d/p) = 1$, $(d/q) = 0$, and $(d/r) = -1$ for every p, q, and r, respectively. Suppose the result to be

$$l = p_1^{\nu_1} \cdots p_i^{\nu_i} \cdot q_1^{\varepsilon_1} \cdots q_j^{\varepsilon_j} \cdot r_1^{\nu_1} \cdots r_k^{\nu_k}.$$

Then the prime factors of the ideal A_1 will be precisely those given by (17), since l^2 is the norm of A_1, and since the unambiguous primes of the first grade enter in pairs of conjugates.

Now determine the groups whose moduli are the prime-power factors of A_1, except that simplicity will be gained by substituting for the pair of groups of types 1 or 2, whose moduli are $P_1^{\nu_1}$ and $(P_1')^{\nu_1}$, respectively, the single group of type 10 or 11, whose modulus is $(P_1 P_1')^{\nu_1}$, and by doing the same in the case of the other P's. Find the generators, therefore, of the groups of types 10 and 11, whose moduli are $(P_1 P_1')^{\nu_1}$, etc., of the groups of types 3–7, whose moduli are $Q_1^{2\varepsilon_1}$, etc., and of the groups of types 8 and 9, whose moduli are $R_1^{\nu_1}$, etc. Finally, find the corresponding generators of the group G, whose modulus is (1), by using the method of § 37. In this way we obtain *a complete set of independent generators of any group of quadratic integers, whose modulus is a rational integer.*

57. *Example* 1. In the field $k(\sqrt{-35})$ find the group G whose modulus is the rational principal ideal $A_1 = (30)$. Here $\omega = \frac{1}{2}(1 + \sqrt{-35})$, $l = 30$, $d = -35$, and the elements of G are residues $a + b\omega$, mod 30, namely those which are prime to 30. Since $30 = 3 \cdot 5 \cdot 2$, and $(d/3) = 1$, $(d/5) = 0$, $(d/2) = -1$, therefore $p = 3$, $q = 5$, and $r = 2$; consequently $(3) = PP'$, $(5) = Q^2$, $(2) = R$, and $(30) = PP' \cdot Q^2 \cdot R$. The group whose modulus is PP' is of type 10 and is generated by -1 and $-1-\omega$, both of period 2: the group whose modulus is Q^2 is of type 3 and is generated by 2, of period 4, and 2ω, of period 5; the group whose modulus is R is of type 9 and is generated by ω, of period 3. The corresponding generators, λ_1, μ_1, λ_2, μ_2, λ_3, of G itself are therefore defined by the congruences

$$\begin{aligned}
\lambda_1 &\equiv -1 && (\text{mod } 3) \equiv 1 && (\text{mod } 5) \equiv 1 && (\text{mod } 2),\\
\mu_1 &\equiv -1-\omega && \text{“} \equiv 1 && \text{“} \equiv 1 && \text{“},\\
\lambda_2 &\equiv 1 && \text{“} \equiv 2 && \text{“} \equiv 1 && \text{“},\\
\mu_2 &\equiv 1 && \text{“} \equiv 2\omega && \text{“} \equiv 1 && \text{“},\\
\lambda_3 &\equiv 1 && \text{“} \equiv 1 && \text{“} \equiv \omega && \text{“}.
\end{aligned}$$

The simplest solutions of these congruences are $\lambda_1 = 11$, $\mu_1 = 11 - 10\omega$, $\lambda_2 = 7$, $\mu_2 = -5 + 12\omega$, $\lambda_3 = 16 + 15\omega$. These are therefore independent generators of G and their periods are 2, 2, 4, 5, and 3, respectively; G is of order 240.

Example 2. By way of strong contrast to the first example take the same modulus (30), but a different field $\mathrm{k}(\sqrt{-1})$. In this case $\omega = \sqrt{-1}$, $d = -4$, $l = 30$, $(d/5) = 1$, $(d/2) = 0$, $(d/3) = -1$, $(5) = PP'$, $(2) = Q^2$, $(3) = R$. Mod PP', the group is of type 10 and is generated by 2 and $1 + \omega$; mod Q^2, the group is of type 7 and is generated by ω; mod R, the group is of type 8 and is generated by $1 + \omega$. G is of order 256, and is generated by $\lambda_1 = 7$, $\mu_1 = 1 + 6\omega$, $\lambda_2 = 16 + 15\omega$, $\lambda_3 = 1 + 10\omega$, whose periods are 4, 4, 2, and 8, respectively.

It is to be noticed that out of the 900 residues $a + b\omega$, mod 30, those which are prime to 30 form two entirely different groups in these two examples, groups which differ in their elements, their orders, their generators, and their invariants. Even residues that are common to the two groups are not necessarily of the same period; e. g., the generator $16 + 15\omega$, which is of period 3 in the first example, is of period 2 in the second.

58. Since completing the above paper I have found in §§ 96–98 of the third volume of Weber's *Algebra*, which was published last year, some results that are rather closely connected with §§ 52–57 of this paper. The immense advantage gained by the theory of numbers from a liberal use of the group concept has been clearly shown by Weber throughout his Algebra (as well as by Bachmann in his books and by G. A. Miller in his memoirs); and although the particular kind of group that I have here investigated is, I believe, entirely new, still Weber approaches very close to it in § 98 of the third volume. His groups O, O', and O_0 are of infinite order, but the quotient-group O/O_0 is of finite order and is a special case of my group G, namely the case in which the modulus is a rational principal ideal; and the quotient-group O'/O_0 is the subgroup of G whose elements have rational residues. Moreover, his "primary ideals," of § 97, are my "completely irrational ideals" of § 52, and his "4. Satz," page 354, is practically the same as my theorem of § 54.

CORNELL UNIVERSITY, *July*, 1909.

A SIMPLIFIED TREATMENT OF THE REGULAR SINGULAR POINT*

BY

GEORGE D. BIRKHOFF

The usual treatment of regular singular points involves the substitution of an infinite series in the differential equation; the series thus formally determined is then proved to be convergent. It is possible to replace this part of the work by a simpler and more direct argument, and this is given below in IV, along with an outline of the entire discussion.† The idea employed is quite similar to that used by LIAPOUNOFF in another connection (see PICARD, *Traité d'Analyse*, vol. 3, pp. 362–363). The analysis is made in full only for a single equation of the second order, and afterward the generalization is indicated.

I. We will assume that it has been proved, say by the method of successive approximations, that the general solution of

$$y'' + p(x)y' + q(x)y = 0, \tag{1}$$

at a point of the x-plane where $p(x)$ and $q(x)$ are single-valued and analytic, has the form

$$y = c_1 y_1 + c_2 y_2,$$

in which y_1 and y_2 are linearly independent and analytic at the point.

II. Now let $x = a$ be an isolated singular point of one or both of the single-valued analytic functions $p(x)$ and $q(x)$, so that by definition $x = a$ is a *singular point of the differential equation.* Consider the solutions y of (1) in a certain vicinity of $x = a$ which is taken so small as to contain no further singular points; these solutions are analytic except perhaps at $x = a$, by I. If we make a positive circuit of $x = a$ the solutions will undergo a linear substitution, which we will write in the form

$$\bar{y}_1 = c_1 y_1 + c_2 y_2,$$
$$\bar{y}_2 = d_1 y_1 + d_2 y_2.$$

There will exist multiplicative solutions, $\bar{y} = \rho y$, for values of ρ such that

$$\begin{vmatrix} c_1 - \rho & d_1 \\ c_2 & d_2 - \rho \end{vmatrix} = 0.$$

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† SCHLESINGER has avoided this substitution (*Journal für die reine und angewandte Mathematik*, vol. 132 (1907), pp. 247–254) but his method is not nearly so simple as the one here presented.

Thus we are led to the conclusion that there are two linearly independent solutions of the form

$$(2)\qquad \begin{aligned} y_1 &= (x-a)^{\kappa_1} f_1(x), \qquad \kappa_1 = \frac{\log \rho_1}{2i\pi}, \\ y_2 &= (x-a)^{\kappa_2} f_2(x), \qquad \kappa_2 = \frac{\log \rho_2}{2i\pi}; \end{aligned}$$

or in a degenerate case, $\rho_1 = \rho_2$, of the form

$$(3)\qquad y_1 = (x-a)^{\kappa_1} f_1(x), \qquad y_2 = y_1 \log (x-a) + (x-a)^{\kappa_1} f_2(x);$$

here f_1 and f_2 are *single-valued* and analytic in the neighborhood of $x = a$, and can therefore be expanded in Laurent series at this point.

III. *Regular solutions* are now defined as solutions of the form (2), (3) in which f_1, f_2 are either analytic or have a pole at $x = a$. A necessary condition that the solutions of (1) be regular at $x = a$ is found at once to be that $p(x)$ and $q(x)$ cannot become infinite to higher than the first and second orders respectively at $x = a$, by writing (1) in the form

$$\frac{1}{\begin{vmatrix} y_1' & y_1 \\ y_2' & y_2 \end{vmatrix}} \begin{vmatrix} y'' & y' & y \\ y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \end{vmatrix} = 0.$$

If the coefficients $p(x)$ and $q(x)$ have this character at $x = a$, that point is called a *regular singular point* of (1).

IV. *Conversely, at a regular singular point the solutions must be regular.* For write

$$z = (x-a)y'.$$

The differential equation (1) has as its equivalent the equations

$$(4)\qquad \begin{aligned} y' &= \frac{z}{x-a}, \\ z' &= (x-a)y'' + y' = -(x-a)[p(x)y' + q(x)y] + y' \\ &= -(x-a)q(x)y + \left(\frac{1}{x-a} - p(x)\right)z. \end{aligned}$$

Consider now the real positive function

$$U = |y^2| + |z^2|$$

for values of x on any ray emanating from $x = a$, and denote the distance $|x-a|$ by r. Since, if $y = \alpha + i\beta$, one has $|y^2| = \alpha^2 + \beta^2$, it is apparent that both terms of U, and therefore U also, are continuous functions of r with continuous r-derivatives. Moreover the inequality

$$\left|\frac{\partial U}{\partial r}\right| \leqq |2yy'| + |2zz'|$$

holds, inasmuch as, for any function w which is analytic, one has

$$w = Re^{\Theta i}, \qquad R = |w|,$$

$$\frac{\partial w}{\partial r} = e^{\Theta i}\left[\frac{\partial R}{\partial r} + iR\frac{\partial \Theta}{\partial r}\right], \qquad w' = \frac{dw}{dx} = \frac{1}{e^{\Theta i}}\frac{\partial w}{\partial r},$$

and hence

$$\left|\frac{\partial |w|}{\partial r}\right| \leqq \left|\frac{\partial w}{\partial r}\right| = |w'|.$$

But in the two equations (4) the coefficients of y and z have poles of at most the first order at $x = a$, so that clearly one has

$$|y'|,\ |z'| \leqq \frac{M}{r}\left\{|y| + |z|\right\},$$

where M is independent of the particular ray under consideration. Accordingly there is obtained

$$\left|\frac{\partial U}{\partial r}\right| \leqq \frac{2M}{r}\left\{|y| + |z|\right\}^2 \leqq \frac{4MU}{r}.$$

This may be written

$$-\frac{4M}{r} \leqq \frac{\partial \log U}{\partial r} \leqq \frac{4M}{r}.$$

Integrating between r and r_0 $(r \leqq r_0)$ one obtains

$$-4M\log\frac{r_0}{r} \leqq \log\frac{U_0}{U} \leqq 4M\log\frac{r_0}{r},$$

so that

$$\left(\frac{r_0}{r}\right)^{-4M} \leqq \frac{U_0}{U} \leqq \left(\frac{r_0}{r}\right)^{4M}.$$

This gives the inequality

$$U \leqq U_0\left(\frac{r}{r_0}\right)^{-4M}.$$

Now U_0 is the value of U at a distance r_0 from $x = a$. Let G denote the upper limit of such values for $0 \leqq \arg(x-a) < 2\pi$. Then we have

$$U \leqq G\left(\frac{r}{r_0}\right)^{-4M}, \qquad [0 \leqq \arg(x-a) < 2\pi].$$

By definition of U it is clear then that

$$|y| \leqq \sqrt{G}\left(\frac{r}{r_0}\right)^{-2M} \qquad [0 \leqq \arg(x-a) < 2\pi],$$

i. e., *the branch of y corresponding to the condition* $0 \leqq \arg(x-a) < 2\pi$ *becomes infinite only to a finite order at* $x=a$.

From equations (2) and (3) it is now concluded that the single-valued functions f_1 and f_2, analytic except perhaps at $x=a$, have also this property. Hence, a positive integer m can be chosen so large that $(x-a)^m f_1$ and $(x-a)^m f_2$ are finite in the neighborhood of $x=a$. By Riemann's theorem these functions are analytic at $x=a$. *Therefore f_1 and f_2 are analytic or have poles at* $x=a$, which we desired to prove.

V. The formal determination of the series (2) and (3) still remains to be made, but for many purposes this expansion is not necessary. The usual discussion shows that there exist essentially two and only two expansions (2) or (3) which formally satisfy (1), and these must, of course, coincide with the desired series.

Part IV of the above treatment (as well as the remainder) admits of extension to a single equation of the nth order in y_1, or n equations of the first order in $y_1, y_2, \cdots, y_n$; in fact the first case is always reduced to the second by a substitution

$$y_2 = (x-a)y_1', \qquad y_3 = (x-a)^2 y_1'', \quad \cdots.$$

If we write the linear system as follows:

$$\frac{dy_i}{dx} = \sum_{j=1}^{n} a_{ij}(x) y_j \qquad (i = 1, 2, \cdots, n), \tag{5}$$

the coefficients $a_{ij}(x)$ have then at most poles of the first order at $x=a$ if $x=a$ is a regular singular point. Hence one finds in a certain vicinity of $x=a$

$$|a_{ij}(x)| \leqq \frac{M}{r}.$$

If one considers the positive real function

$$U = \sum_{i=1}^{n} |y_i^2|$$

one finds

$$\begin{aligned} \left|\frac{\partial U}{\partial r}\right| &\leqq \sum_{i=1}^{n} |2y_i y_i'| \\ &\leqq \frac{2M}{r}\left(\sum_{i=1}^{n} |y_i|\right)^2. \qquad \text{by (5),} \\ &\leqq \frac{2nM}{r} U. \end{aligned}$$

The remainder of the argument is now made as before.

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THE STRAIN OF A GRAVITATING, COMPRESSIBLE ELASTIC SPHERE*

BY

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In the solution of the problem of the strain of a homogeneous elastic sphere, as given by LAMÉ and by KELVIN, the bodily forces are supposed expressible as known functions of the coördinates of position. When self-gravitation is among the forces considered, however, the bodily forces depend in part upon the state of strain, and it is only in the case of assumed incompressibility that the usual method of solution is applicable.

The present paper has for its main object the solution of the problem of the strain of an isotropic elastic sphere, initially homogeneous, due to disturbing forces of a certain type, taking into account the changes in the gravitational forces which result from the strain.

The contents of the paper fall under the following four heads:

I. Strain of a gravitating, compressible elastic sphere under the action of tidal or centrifugal forces, with numerical computations for the case in which the surface is free from stress.

II. Strain of a gravitating, compressible elastic sphere covered by a shallow ocean, under the action of tidal or centrifugal forces.

III. Effect of compressibility upon estimates of the rigidity of the earth.

IV. Strain of a gravitating, compressible elastic sphere under the action of small disturbing forces derivable from a potential which is any spherical harmonic of degree not less than 2.

I. STRAIN OF A COMPRESSIBLE, GRAVITATING ELASTIC SPHERE UNDER THE ACTION OF TIDAL OR CENTRIFUGAL FORCES.

§ 1. *Physical analysis of the problem.*

The object of the analysis is to determine the strain produced in an elastic sphere by the action of disturbing forces of the type of tidal or centrifugal forces — i. e., forces derivable from a potential proportional to $P_2 r^2$, in which P_2 is the zonal surface harmonic $\frac{3}{2}\cos^2\theta - \frac{1}{2}$. The disturbing forces are assumed small in comparison with the actual forces of gravity, but the changes in the gravita-

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tional forces due to the strain are assumed to be of the same order as the disturbing forces. This accords with the facts in the case of the actual tidal and centrifugal forces acting upon the earth.

Before the action of the disturbing forces the body is assumed to be in equilibrium under the action of the mutual gravitation of its parts without surface stress. The stress-condition at any interior point need not be specified more definitely; it may be hydrostatic equilibrium, or any departure from that condition which is consistent with the forces of self-attraction. The stresses and strains considered in the following analysis are to be regarded as superposed upon the supposed initial condition, it being assumed that small departures from this initial condition follow the laws of elasticity as ordinarily given for a homogeneous isotropic solid.*

When the external disturbing forces come into action, the body assumes a new configuration of equilibrium. The gravitational force which now acts upon a given element of volume is different from that which originally acted upon the same element, for two reasons, — because the density of the attracted element is changed, and because the density-distribution of the attracting mass is changed. In computing the new value of the gravitational force the new configuration of the body may be treated as made up of two parts, — the original configuration, and a certain mass-distribution superposed upon it. The total attraction upon a volume-element may then be expressed as the resultant of the following parts: (1) The attraction of the original mass-configuration upon the original mass of the element; (2) the attraction of the original mass-configuration upon the increment of mass of the element; (3) the attraction of the increment of mass-distribution upon the original mass of the element; and (4) the attraction of the increment of mass-distribution upon the increment of mass of the element. Of these four forces (1) is balanced by the initial stresses and may be omitted from the equations of equilibrium of the element; (2) and (3) are of the same order of magnitude as the disturbing forces; while (4) is negligible in comparison with (2) and (3). Hence (2) and (3) are the only bodily forces to be considered in addition to the external disturbance.

* It has been held by certain writers that the existence of initial stresses of great magnitude vitiates the application of the theory of elasticity to a gravitating body comparable in size and mass with the earth. As to applications which are sometimes made of the theory, the objection appears to be undoubtedly valid. For example, it would be absurd to compute the actual stresses within the earth as if they were due to gravitation acting upon a body initially free from stress; even if such an assumption were in harmony with any reasonable supposition as to the history of the earth, the strains thus computed would far surpass the limits of elasticity. This objection does not, however, appear to hold against such an application of the laws of elasticity as is made in the present paper. To assume that a body under great stress, whether this is everywhere purely a normal pressure or whether involving also tangential stresses, is elastic for small departures from the initial configuration, is not inconsistent with what is known regarding bodies under ordinary experimental conditions. It can hardly be doubted that the great body of the earth is elastic as regards slight changes of configuration.

The original density being called ρ, the increment of density $\delta\rho$ is equal to $-\rho\Delta$, if Δ is the cubical expansion. Hence the force (2) is to be expressed as the attraction of the undisturbed mass upon a density $-\rho\Delta$, while (3) is the attraction of a mass-distribution of density $-\rho\Delta$ upon a density ρ, together with the attraction due to the surface inequality produced by the strain.

Besides the bodily forces it is necessary to consider the stress-condition at the surface. Upon the actual bounding surface it is assumed that the stress remains zero; but if (as is convenient) the equations of interior equilibrium be assumed to apply throughout the original spherical volume, the surface of this volume must be regarded as sustaining a stress whose value must be properly expressed. It is shown below that, to the first order of small quantities, this stress is radial and equal to the weight of the surface inequality produced by the strain.

§ 2. *Formation of equations of equilibrium for an element of volume.*

(*a*) *General form of equations in terms of polar coördinates.* Let the axis of symmetry of the disturbing forces be taken as initial line of a system of polar coördinates, the displacements in the three primary directions at a point (r, θ, ϕ) being called

$$u_r, \qquad u_\theta, \qquad u_\phi,$$

while Δ is the cubical expansion. In the case now under consideration

$$u_\phi = 0.$$

Let the total bodily force per unit volume have components

$$\rho F_r, \qquad \rho F_\theta,$$

(the ϕ-component being 0).

In accordance with a common notation, let μ denote the modulus of distortion and $\lambda + \frac{2}{3}\mu$ the bulk-modulus of the material. The equations of equilibrium for an elementary volume then assume the following forms:*

$$(1) \qquad (\lambda+2\mu)\frac{\partial\Delta}{\partial r} - \frac{\mu}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right)\sin\theta\right] + \rho F_r = 0,$$

$$(2) \qquad (\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta} + \frac{\mu}{r\sin\theta}\frac{\partial}{\partial r}\left[\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right)\sin\theta\right] + \rho F_\theta = 0;$$

in which

$$(3) \qquad \Delta = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2 u_r \sin\theta) + \frac{\partial}{\partial\theta}(ru_\theta\sin\theta)\right].$$

(*b*) *Assumptions.* It is found that by making certain assumptions the equations are reduced to a form admitting of solution. It will be well to state these

*See A. E. H. Love's *A Treatise on the Mathematical Theory of Elasticity*, 2d edition (1906), pp. 138, 56. The terms representing bodily forces are omitted from the equations as given on p. 138.

assumptions explicitly, at the same time remarking that their justification is the fact that they lead to a solution satisfying all the conditions of the problem.

(1) The potential of the disturbing forces being proportional to $r^2 P_2$, in which $P_2 = \frac{3}{2}\cos^2\theta - \frac{1}{2}$, it is assumed that the radial displacement and the cubical expansion depend upon θ in the same way as this potential does; i. e., each is assumed to be the product of P_2 by a function of r.

(2) It is assumed that the mean radius of an originally spherical surface is unchanged by the strain.

(c) *Linear displacements replaced by ellipticity and maximum angular displacement.* It is found convenient to replace u_r and u_θ in the foregoing equations by two quantities e, α, defined as follows:

The former represents the ellipticity of the surface originally spherical and of radius r; that is,

$$u_r = -\tfrac{2}{3} e r P_2 \tag{4}$$

(it being assumed that a positive value of e corresponds to oblateness).

The quantity α represents the angular displacement of a radius vector for which $\theta = 45°$. It is then found that, for any point (r, θ),

$$\frac{u_\theta}{r} = \alpha \sin 2\theta. \tag{5}$$

This follows from the assumption that Δ involves θ only in a factor P_2.

To make the last statement clear, notice that the introduction of e in place of u_r reduces (3) to the form

$$\Delta = -\left(2e + \tfrac{2}{3} r \frac{de}{dr}\right) P_2 + \frac{1}{r \sin\theta} \frac{\partial (u_\theta \sin\theta)}{\partial \theta}.$$

In order that the last term may be equivalent to P_2 multiplied into a function of r, it is easy to show that we must have

$$u_\theta = \alpha \sin 2\theta + \frac{\beta}{\sin\theta},$$

in which α and β are independent of θ. Infinite values of u_θ being excluded, β must be 0.*

If, now, u_r and u_θ be replaced by e and α by means of (4) and (5), equations (1) and (2) take the following forms:

* It is otherwise evident that, for a given value of r, u_θ will have equal values for complementary values of θ. This is seen by considering the effect of superposing strains of the same type about different axes. Thus the θ-displacements in a given meridian plane (say that of xz) will be annulled by superposing upon the given strain either (a) an exactly similar strain symmetrical about the x-axis, or (b) a strain symmetrical about the z-axis which is just the reverse of the given strain.

$$(6)\quad \left[4(\lambda+\mu)\frac{d\alpha}{dr}-8\mu\frac{\alpha}{r}-\tfrac{2}{3}(\lambda+2\mu)r\frac{d^2e}{dr^2}-\tfrac{8}{3}(\lambda+2\mu)\frac{de}{dr}+4\mu\frac{e}{r}\right]P_2+\rho F_r=0,$$

$$(7)\quad \left[\mu r\frac{d^2\alpha}{dr^2}+4\mu\frac{d\alpha}{dr}-2(3\lambda+5\mu)\frac{\alpha}{r}+(\lambda+\mu)\frac{de}{dr}+(3\lambda+5\mu)\frac{e}{r}\right]\sin 2\theta+\rho F_\theta=0,$$

while the value of Δ becomes

$$(8)\quad \Delta=\left(4\alpha-2e-\tfrac{2}{3}r\frac{de}{dr}\right)P_2.$$

(*d*) *Expression of bodily forces.* The bodily forces acting upon an elementary volume are (1) the disturbing force (here supposed to be the centrifugal force due to a uniform rotation about a fixed axis), (2) the gravitational force due to the change of density of the attracted element, (3) the attraction due to the change of configuration of the attracting mass. We proceed to get expressions for these in terms of r, θ, e and α.

(1) The disturbing force.

Representing the angular velocity of the rotation by ω, the centrifugal forces have a potential

$$(9)\quad W=-\tfrac{1}{3}\omega^2r^2P_2$$

(neglecting a term independent of θ). If a is the radius of the sphere, g the value of gravity at the surface, and c the ratio of equatorial centrifugal force to gravity, i. e.,

$$(10)\quad c=\frac{\omega^2a}{g},$$

we may write

$$(11)\quad W=-\frac{cg}{3a}r^2P_2.$$

The parts of F_r and F_θ corresponding to this potential are

$$(12)\quad \frac{\partial W}{\partial r}=-\frac{2cg}{3a}rP_2,$$

$$(13)\quad \frac{1}{r}\frac{\partial W}{\partial\theta}=\frac{cg}{2a}r\sin 2\theta.$$

(2) Force due to change of density of attracted element.

The attraction per unit mass at any point, due to the original homogeneous sphere, is a radial force equal to

$$-g\frac{r}{a}.$$

The increment of density of the attracted element being $-\rho\Delta$, the force per unit volume is

$$\left(-g\frac{r}{a}\right)(-\rho\Delta)=g\frac{\rho r\Delta}{a};$$

hence this force is accounted for by adding to the value of ρF_r a term

$$\frac{\rho g r \Delta}{a} = \rho g \frac{r}{a}\left(4\alpha - 2e - \tfrac{2}{3}r\frac{de}{dr}\right)P_2. \tag{14}$$

(3) Force due to change of configuration of the attracting mass.

We have now to express the value of the potential at any point due to a mass-distribution of density $-\rho\Delta$ throughout the original spherical volume, together with that due to the surface layer superposed upon this sphere by reason of the strain.

Consider a shell of radius r', thickness dr', and density

$$-\rho\Delta' = -\rho\left(4\alpha' - 2e' - \tfrac{2}{3}r'\frac{de'}{dr'}\right)P_2.$$

The potential at a point (r, θ) due to this shell is

$$-\tfrac{4}{5}\pi\gamma\rho\left(4\alpha' - 2e' - \tfrac{2}{3}r'\frac{de'}{dr'}\right)\frac{r^2}{r'}\,dr'\cdot P_2 \qquad (\text{if } r<r'), \tag{15}$$

$$-\tfrac{4}{5}\pi\gamma\rho\left(4\alpha' - 2e' - \tfrac{2}{3}r'\frac{de'}{dr'}\right)\frac{r'^4}{r^3}\,dr'\cdot P_2 \qquad (\text{if } r>r'). \tag{16}$$

Hence the potential at any point (r, θ) within the spherical volume of radius a, due to the density-distribution $-\rho\Delta$, is

$$-\frac{4\pi\gamma\rho P_2}{5}\left[\frac{1}{r^3}\int_0^r\left(4\alpha' - 2e' - \tfrac{2}{3}r'\frac{de'}{dr'}\right)r'^4dr' + r^2\int_r^a\left(4\alpha' - 2e' - \tfrac{2}{3}r'\frac{de'}{dr'}\right)\frac{dr'}{r'}\right]. \tag{17}$$

To this must be added the potential due to the surface inequality. If e_1 is the surface value of e, the thickness of the layer is $-\frac{2}{3}e_1 aP_2$, and the potential due to it at a point (r, θ) within the body is

$$-\tfrac{8}{15}\pi\gamma\rho e_1 r^2 P_2. \tag{18}$$

The total potential due to the change of mass-distribution may therefore be written in the following form, in which g replaces $\frac{4}{3}\pi\gamma\rho a$, the surface value of gravity:

$$V = -\frac{3gP_2}{5a}\left[\frac{1}{r^3}\int_0^r\left(4\alpha' - 2e' - \tfrac{2}{3}r'\frac{de'}{dr'}\right)r'^4dr' + r^2\int_r^a\left(4\alpha' - 2e' - \tfrac{2}{3}r'\frac{de'}{dr'}\right)\frac{dr'}{r'} + \tfrac{2}{3}e_1 r^2\right]. \tag{19}$$

By partial integration we find

$$\frac{1}{r^3}\int_0^r\left(4\alpha' - 2e' - \tfrac{2}{3}r'\frac{de'}{dr'}\right)r'^4\,dr' = \frac{4}{3r^3}\int_0^r(3\alpha' + e')r'^4\,dr' - \tfrac{2}{3}r^2 e,$$

$$r^2\int_r^a\left(4\alpha'-2e'-\tfrac{2}{3}r'\frac{de'}{dr'}\right)\frac{dr'}{r'}=2r^2\int_r^a(2\alpha'-e')\frac{dr'}{r'}+\tfrac{2}{3}r^2(e-e_1),$$

so that the expression for V reduces to

$$V=-\frac{2gP_2}{5a}\left[\frac{2}{r^3}\int_0^r(3\alpha'+e')r'^4\,dr'+3r^2\int_r^a(2\alpha'-e')\frac{dr'}{r'}\right]. \tag{20}$$

The forces derived from the potential V are (per unit mass)

$$\frac{\partial V}{\partial r}=\frac{12gP_2}{5a}\left[\frac{1}{r^4}\int_0^r(3\alpha'+e')r'^4\,dr'-r\int_r^a(2\alpha'-e')\frac{dr'}{r'}-\tfrac{5}{6}er\right], \tag{21}$$

$$\frac{1}{r}\frac{\partial V}{\partial\theta}=\frac{3g\sin 2\theta}{5a}\left[\frac{2}{r^4}\int_0^r(3\alpha'+e')r'^4\,dr'+3r\int_r^a(2\alpha'-e')\frac{dr'}{r'}\right]. \tag{22}$$

(4) Components of total bodily force.

Combining the foregoing results, we have for the components of the bodily force per unit volume at the point (r, θ),

$$\rho F_r=\frac{\rho gP_2}{a}\left[-\tfrac{2}{3}cr+\left(4\alpha-4e-\tfrac{2}{3}r\frac{de}{dr}\right)r+\frac{12}{5r^4}\int_0^r(3\alpha'+e')r'^4dr'-\frac{12r}{5}\int_r^a(2\alpha'-e')\frac{dr'}{r'}\right], \tag{23}$$

$$\rho F_\theta=\frac{\rho g\sin 2\theta}{a}\left[\tfrac{1}{2}cr+\frac{6}{5r^4}\int_0^r(3\alpha'+e')r'^4dr'+\frac{9r}{5}\int_r^a(2\alpha'-e')\frac{dr'}{r'}\right]. \tag{24}$$

(e) *Final form of equations of equilibrium for elementary volume.* In writing the two equations of equilibrium for a volume element in their final form, it is convenient to introduce the factor r throughout each equation, and then to replace r as independent variable by $x=r/a$. The equations then take the following forms:

$$4(\lambda+\mu)x\frac{d\alpha}{dx}-8\mu\alpha-\tfrac{2}{3}(\lambda+2\mu)x^2\frac{d^2e}{dx^2}-\tfrac{8}{3}(\lambda+2\mu)x\frac{de}{dx}+4\mu e+\rho ga\left[-\tfrac{2}{3}cx^2+\left(4\alpha-4e-\tfrac{2}{3}x\frac{de}{dx}\right)x^2+\frac{12}{5x^3}\int_0^x(3\alpha'+e')x'^4dx'-\frac{12x^2}{5}\int_x^1(2\alpha'-e')\frac{dx'}{x'}\right]=0, \tag{25}$$

$$\mu x^2\frac{d^2\alpha}{dx^2}+4\mu x\frac{d\alpha}{dx}-2(3\lambda+5\mu)\alpha+(\lambda+\mu)x\frac{de}{dx}+(3\lambda+5\mu)e+\rho ga\left[\tfrac{1}{2}cx^2+\frac{6}{5x^3}\int_0^x(3\alpha'+e')x'^4dx'+\frac{9x^2}{5}\int_x^1(2\alpha'-e')\frac{dx'}{x'}\right]=0. \tag{26}$$

§ 3. *Surface conditions.*

Upon the bounding surface of the strained body the stress is zero. It seems most convenient, however, to treat the strained body as extending to or beyond the original spherical boundary $r = a$ in every direction, and to determine what stress must exist upon this surface in order that the surface $r = a(1 - \frac{2}{3}e_1 P_2)$ may be free from stress. For this purpose consider the layer of matter of density ρ included between the surfaces $r = a$, $r = a(1 - \frac{2}{3}e_1 P_2)$. Let equations of equilibrium be formed for an elementary portion of this layer bounded by a meridian plane and a plane parallel to it at an infinitesimal distance h, and by two radial planes perpendicular to the meridian inclined to each other at an infinitesimal angle ψ; the section of such an element by the meridian plane is represented by $ABB'A'$ (Fig. 1).

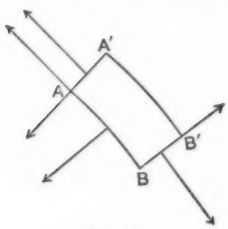

FIG. 1.

We have

$$AB = a\psi, \qquad AA' = -\tfrac{2}{3}ae_1 P_2,$$

$$BB' = AA' + \psi \frac{\partial}{\partial\theta}(-\tfrac{2}{3}ae_1 P_2) = AA' + ae_1 \psi \sin 2\theta.$$

The stresses per unit area at the point A are

$$\widehat{rr} = \lambda\Delta + 2\mu \frac{\partial u_r}{\partial r} = \left[4\lambda\alpha - 2(\lambda + \tfrac{2}{3}\mu)e - \tfrac{2}{3}(\lambda + 2\mu)x \frac{de}{dx} \right] P_2, \tag{27}$$

$$\widehat{\theta\theta} = \lambda\Delta + 2\mu\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)$$

$$= \left[4\lambda\alpha - 2(\lambda + \tfrac{2}{3}\mu)e - \tfrac{2}{3}\lambda x \frac{de}{dx} \right] P_2 + 4\mu\alpha \cos 2\theta, \tag{28}$$

$$\widehat{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) = \mu\left(e + x\frac{d\alpha}{dx}\right)\sin 2\theta. \tag{29}$$

From these expressions values may be found for the total normal and tangential forces on the faces AB, AA', BB' of the element, while the total bodily force is the weight of the element, which is equal to ρg into its volume. By writing the two equations of equilibrium for the element, and retaining only terms of lowest order in the small quantities e, α, it is found that

$$-\widehat{rr} + \rho g(\tfrac{2}{3}eaP_2) = 0, \tag{30}$$

$$\widehat{r\theta} = 0. \tag{31}$$

The equations to be satisfied at the surface $r = a$ are therefore

$$(32) \qquad \frac{\widehat{rr}}{P_2} = 4\lambda\alpha - 2(\lambda + \tfrac{2}{3}\mu)e - \tfrac{2}{3}(\lambda + 2\mu)x\frac{de}{dx} = \tfrac{2}{3}\rho g a e,$$

$$(33) \qquad \frac{\widehat{r\theta}}{\sin 2\theta} = \mu\left(e + x\frac{d\alpha}{dx}\right) = 0.$$

§ 4. *Preliminary discussion of special cases.*

The problem is now reduced to that of solving the simultaneous ordinary differential equations (25) and (26), subject to the conditions expressed by (32) and (33). Before proceeding to the general solution, it is of interest to note the form assumed by the equations in special cases for which the solutions are known. Such cases are the following:

(*A*) A non-gravitating body acted upon by bodily disturbing forces having a potential proportional to $P_2 r^2$ (or free from bodily forces), and

(1) With any given surface stresses of type corresponding to the bodily forces (i. e., the radial stress proportional to P_2 and the tangential stress to $\sin 2\theta$), or

(2) With any given surface displacements of the same type (i. e., the radial displacement proportional to P_2 and the tangential to $\sin 2\theta$).

(*B*) A gravitating incompressible body with bodily and surface forces and surface displacements as in any of the cases under (*A*).

Of these cases the complete solution will here be given only for that of a non-gravitating body with surface free from stress. The modifications in the analysis required by the other cases under (*A*) are sufficiently obvious, while the corresponding case of (*B*) is given later on as a limiting case of the general solution for a gravitating compressible body.

It may be remarked, also, that after the general solution has been given it will be obvious how it must be modified if the surface conditions are changed.

(*a*) *Solution for the case of a non-gravitating body acted upon by bodily forces but free from surface forces.* In this case all the terms in equations (25) and (26) which represent bodily forces disappear except those representing the disturbing forces, and the equations become

$$(34) \quad 4(\lambda+\mu)x\frac{d\alpha}{dx} - 8\mu\alpha - \tfrac{2}{3}(\lambda+2\mu)x^2\frac{d^2e}{dx^2} - \tfrac{5}{3}(\lambda+2\mu)x\frac{de}{dx} + 4\mu e - \tfrac{2}{3}\rho g a c x^2 = 0,$$

$$(35) \quad \mu x^2\frac{d^2\alpha}{dx^2} + 4\mu x\frac{d\alpha}{dx} - 2(3\lambda+5\mu)\alpha + (\lambda+\mu)x\frac{de}{dx} + (3\lambda+5\mu)e + \tfrac{1}{2}\rho g a c x^2 = 0.$$

The surface conditions (32) and (33) become

$$(36) \quad 4\lambda\alpha - 2(\lambda + \tfrac{2}{3}\mu)e - \tfrac{2}{3}(\lambda + 2\mu)x\frac{de}{dx} = 0, \qquad e + x\frac{d\alpha}{dx} = 0 \quad (x=1).$$

It is easily shown that these equations may all be satisfied by values of e and α of the forms

$$e = A_0 + A_2 x^2, \tag{37}$$

$$\alpha = B_0 + B_2 x^2, \tag{38}$$

by properly determining the constants A_0, A_2, B_0, B_2.

To satisfy the differential equations we must have

$$B_0 = \tfrac{1}{2} A_0, \tag{39}$$

while A_2 and B_2 must satisfy the two equations

$$\begin{aligned} 8\lambda B_2 - \tfrac{4}{3}(5\lambda + 7\mu) A_2 - \tfrac{2}{3}\rho g a c = 0, \\ -6\lambda B_2 + (5\lambda + 7\mu) A_2 + \tfrac{1}{2}\rho g a c = 0, \end{aligned} \tag{40}$$

which are seen to be identical. The surface conditions give two additional relations among the four constants; these are

$$4\lambda(B_0 + B_2) - 2(\lambda + \tfrac{2}{3}\mu)(A_0 + A_2) - \tfrac{4}{3}(\lambda + 2\mu) A_2 = 0, \tag{41}$$

$$A_0 + A_2 + 2B_2 = 0. \tag{42}$$

Solving these equations, we have

$$A_0 = \frac{4\lambda + 3\mu}{19\lambda + 14\mu} \frac{\rho g a}{\mu} c, \qquad A_2 = -\frac{3\lambda + 2\mu}{2(19\lambda + 14\mu)} \frac{\rho g a}{\mu} c,$$

$$B_0 = \frac{4\lambda + 3\mu}{2(19\lambda + 14\mu)} \frac{\rho g a}{\mu} c, \qquad B_2 = -\frac{5\lambda + 4\mu}{4(19\lambda + 14\mu)} \frac{\rho g a}{\mu} c.$$

Hence, finally, the values

$$e = \frac{c}{2(19\lambda + 14\mu)} [2(4\lambda + 3\mu) - (3\lambda + 2\mu) x^2] \frac{\rho g a}{\mu}, \tag{43}$$

$$\alpha = \frac{c}{4(19\lambda + 14\mu)} [2(4\lambda + 3\mu) - (5\lambda + 4\mu) x^2] \frac{\rho g a}{\mu}, \tag{44}$$

which agree with the known solution for this case.

(*b*) *Non-gravitating body with given surface forces or surface displacements.* If the two surface stresses are not zero but are known functions of θ of the type above specified, the last members of (32) and (33) become known constants. If the surface displacements are given and of the type specified, (32) and (33) are replaced by equations obtained in an obvious manner.

(*c*) *Non-gravitating spherical shell.* The solution for a spherical shell may be obtained by assuming

$$e = A_{-5} x^{-5} + A_{-3} x^{-3} + A_0 + A_2 x^2, \tag{45}$$

$$\alpha = B_{-5} x^{-5} + B_{-3} x^{-3} + B_0 + B_2 x^2, \tag{46}$$

the constants being determined to satisfy the conditions at the inner and outer boundaries. It may in fact be shown that these equations satisfy (34) and (35) if A_0, B_0, A_2, B_2 satisfy (39) and (40), and the remaining coefficients the conditions

$$\mu A_{-3} - (3\lambda + 5\mu) B_{-3} = 0, \tag{47}$$

$$A_{-5} + 3B_{-5} = 0. \tag{48}$$

There are thus four constants available for satisfying the boundary conditions.

(*d*) *Incompressible gravitating body.* In this case all the gravitational forces which are due to changes of density vanish, but the attraction of the surface inequality remains. The potential due to this surface layer has already been given, being

$$\tfrac{8}{15}\pi\gamma\rho e_1 r^2 P_2 = -\frac{2g}{5a} e_1 r^2 P_2.$$

This gives in equations (25) and (26) the terms

$$-\tfrac{4}{5}\rho g a e_1 x^2 \qquad \text{and} \qquad \tfrac{3}{5}\rho g a e_1 x^2.$$

Since e_1 is a constant, the solution may proceed as in case (*a*), by assuming

$$e = A_0 + A_2 x^2, \tag{49}$$

$$\alpha = B_0 + B_2 x^2. \tag{50}$$

In satisfying the equations which determine the constants, special reasoning is required by the fact that λ is infinite. This presents no serious difficulty, however, and will not be here given, since the case of incompressibility is hereinafter treated as a special case of the general problem of a gravitating sphere. To the solution of the general problem we now proceed.

§ 5. *Solution of equations in the general case.*

Inspection of the differential equations (25) and (26), and the form of the solution in the special cases above considered, suggest the trial of power series for the values of e and α.

The substitution of

$$e = A_m x^m, \qquad \alpha = B_m x^m \tag{51}$$

results in the following terms in the two equations: In (25),

$$\begin{aligned} &\big(\big[4(\lambda+\mu)m - 8\mu\big]B_m - \big[\tfrac{2}{3}(\lambda+2\mu)(m^2+3m) - 4\mu\big]A_m\big)x^m \\ &+ \rho g a\Big[\frac{2(m^2+8m+6)}{3m(m+5)}(6B_m - (m+3)A_m)x^{m+2} \\ &\qquad - \frac{12}{5m}(2B_m - A_m)x^2\Big]. \end{aligned} \tag{52}$$

In (26),

$$\left([\mu(m^2+3m)-2(3\lambda+5\mu)]B_m+[(\lambda+\mu)m+(3\lambda+5\mu)]A_m\right)x^m$$
$$+\rho g a\left[-\frac{3}{m(m+5)}(6B_m-(m+3)A_m)x^{m+2}\right. \tag{53}$$
$$\left.+\frac{9}{5m}(2B_m-A_m)x^2\right].$$

Notice first that the coefficients of x^2 in these two expressions are in a constant ratio independent of m; hence if m be given a series of values, the coefficients of x^2 in the two equations may be made to vanish by a single relation among the coefficients.

Again, notice that terms in x^{m+2} result only from the following terms in the values of e and α:

$$A_m x^m + A_{m+2}x^{m+2}, \qquad B_m x^m + B_{m+2}x^{m+2};$$

and that the coefficients of x^{m+2} in both equations will vanish if the following two equations are satisfied:

$$[4(\lambda+\mu)m+8\lambda]B_{m+2}-[\tfrac{2}{3}(\lambda+2\mu)m(m+7)+\tfrac{4}{3}(5\lambda+7\mu)]A_{m+2}$$
$$+\rho g a\frac{2(m^2+8m+6)}{3m(m+5)}[6B_m-(m+3)A_m]=0, \tag{54}$$

$$[\mu m(m+7)-6\lambda]B_{m+2}+[(\lambda+\mu)m+5\lambda+7\mu]A_{m+2}$$
$$-\rho g a\frac{3}{m(m+5)}[6B_m-(m+3)A_m]=0. \tag{55}$$

Finally, notice that the terms of lowest degree in the series for e and α will introduce into the two equations terms of like degree which must vanish identically. That is, the terms

$$A_m x^m, \quad B_m x^m$$

of lowest degree must be such that the coefficients of x^m in (52) and (53) will vanish. The values of m for which this is possible without the vanishing of A_m and B_m are found by equating to zero the determinant of the coefficients of A_m and B_m in these expressions; this determinant reduces to

$$\tfrac{2}{3}\mu(\lambda+2\mu)(m-2)m(m+3)(m+5).$$

The lowest power of x in the two series must therefore be either

$$x^2, \ x^0, \ x^{-3} \text{ or } x^{-5}.$$

Since negative powers of x are excluded by physical considerations (because the solution must apply to the point $x=0$), we conclude that the terms available

for the solution constitute ascending series of even powers of x beginning with the power zero. It is easily found also that we must have * $B_0 = \frac{1}{2}A_0$.

We now assume

$$e = A_0 + A_2x^2 + A_4x^4 + \cdots, \tag{56}$$

$$\alpha = \tfrac{1}{2}A_0 + B_2x^2 + B_4x^4 + \cdots, \tag{57}$$

and determine the relations which must hold among the coefficients in order that the coefficient of every power of x in each of two equations may vanish identically.

From the above discussion it is evident that if $m > 0$, the equations to be satisfied are (54) and (55), while the vanishing of the coefficient of x^2 requires that

$$8\lambda B_2 - \tfrac{4}{3}(5\lambda + 7\mu)A_2 - g\rho a\left[\tfrac{4}{5}A_0 + \frac{12}{5}\sum_{m=2}^{\infty}\frac{1}{m}(2B_m - A_m)\right] = \tfrac{2}{3}\rho g a c \tag{58}$$

(odd values of m being excluded in the summation).

The solution of equations (54) and (55) gives

$$A_{m+2} = \frac{\mu m^2 + 10\mu m - 6(\lambda - 2\mu)}{\mu m(m+2)(m+5)(m+7)}[6B_m - (m+3)A_m]\frac{g\rho}{\lambda + 2\mu}, \tag{59}$$

$$B_{m+2} = -\frac{(\lambda+\mu)m + 5\lambda + 4\mu}{\mu m(m+2)(m+5)(m+7)}[6B_m - (m+3)A_m]\frac{g\rho}{\lambda + 2\mu}, \tag{60}$$

$$6B_{m+2} - (m+5)A_{m+2} = -\frac{m+6}{m(m+5)}[6B_m - (m+3)A_m]\frac{g\rho}{\lambda + 2\mu}. \tag{61}$$

By successive applications of the last equation may be derived the following values of the coefficients of x^4 and higher powers, n having values 0, 1, 2, $\cdots$:

$$A_{4+2n} = -2\tfrac{1}{4}(6B_2 - 5A_2)\left(-\frac{g\rho}{\lambda+2\mu}\right)^{n+1}\left[4(n^2+7n+9) - 6\frac{\lambda}{\mu}\right]\frac{n+3}{7\cdot 9\cdots(2n+9)}, \tag{62}$$

$$B_{4+2n} = 2\tfrac{1}{4}(6B_2 - 5A_2)\left(-\frac{g\rho}{\lambda+2\mu}\right)^{n+1}\left[2(n+3) + (2n+7)\frac{\lambda}{\mu}\right]\frac{n+3}{7\cdot 9\cdots(2n+9)}, \tag{63}$$

so that the values of e and α may be written

$$\begin{aligned} e = {} & A_0 + A_2x^2 + 2\tfrac{1}{4}(6B_2 - 5A_2)\frac{g\rho}{\lambda + 2\mu}x^4 \\ & \times \sum_0^{\infty}(-1)^n\left[4(n^2+7n+9) - 6\frac{\lambda}{\mu}\right]\frac{n+3}{7\cdot 9\cdots(2n+9)}\left(\frac{g\rho}{\lambda+2\mu}x^2\right)^n, \end{aligned} \tag{64}$$

* This is evident algebraically. It is also evident otherwise that if the solution has the assumed form, B_0 must equal $\frac{1}{2}A_0$; for A_0 and B_0 are the values of e and α at the center, where these quantities will be related as in the case of a homogeneous strain.

$$\alpha = \tfrac{1}{2}A_0 + B_2x^2 - \tfrac{1}{24}(6B_2 - 5A_2)\frac{ga\rho}{\lambda + 2\mu}x^4$$

(65)
$$\times \sum_0^\infty (-1)^n \left[2(n+3) + (2n+7)\frac{\lambda}{\mu}\right] \frac{n+3}{7\cdot 9\cdots(2n+9)} \left(\frac{ga\rho}{\lambda+2\mu}x^2\right)^n,$$

in which A_0, A_2, B_2 must satisfy (58) and the surface conditions (32) and (33).

For substitution in (58) we find (for $n = 0, 1, 2, \cdots$)

$$\frac{2B_{4+2n} - A_{4+2n}}{4+2n} = \tfrac{1}{12}(6B_2 - 5A_2)\left(-\frac{ga\rho}{\lambda+2\mu}\right)^{n+1}$$

(66)
$$\times \left(n + 6 + \frac{\lambda}{\mu}\right) \frac{n+3}{7\cdot 9\cdots(2n+9)}.$$

In order to express as concisely as possible the values of e and α and the three equations to determine the constants, the following notation may be employed:

(67) $$L(z) = \sum_0^\infty (-1)^n \left[4(n^2+7n+9) - 6\frac{\lambda}{\mu}\right] \frac{n+3}{7\cdot 9\cdots(2n+9)} z^n,$$

(68) $$M(z) = \sum_0^\infty (-1)^n \left[2(n+3) + (2n+7)\frac{\lambda}{\mu}\right] \frac{n+3}{7\cdot 9\cdots(2n+9)} z^n,$$

(69) $$N(z) = \sum_0^\infty (-1)^n \left[n + 6 + \frac{\lambda}{\mu}\right] \frac{n+3}{7\cdot 9\cdots(2n+9)} z^n,$$

(70) $$L'(z) = \sum_0^\infty (-1)^n \left[4(n^2+7n+9) - 6\frac{\lambda}{\mu}\right] \frac{(n+2)(n+3)}{7\cdot 9\cdots(2n+9)} z^n,$$

(71) $$M'(z) = \sum_0^\infty (-1)^n \left[2(n+3) + (2n+7)\frac{\lambda}{\mu}\right] \frac{(n+2)(n+3)}{7\cdot 9\cdots(2n+9)} z^n,$$

(72) $$b = \frac{ga\rho}{\lambda + 2\mu}.$$

The solution then takes the following form, the last two of the five equations being derived from the surface conditions (32) and (33):

(*A*) $$e = A_0 + A_2x^2 + \tfrac{1}{24}(6B_2 - 5A_2)bx^4 L(bx^2),$$

(*B*) $$\alpha = \tfrac{1}{2}A_0 + B_2x^2 - \tfrac{1}{24}(6B_2 - 5A_2)bx^4 M(bx^2),$$

(*C*)
$$4(A_0 + A_2) + \frac{140}{3b(\lambda/\mu + 2)}A_2$$
$$+ (6B_2 - 5A_2)\left(2 - \frac{20\lambda/\mu}{3b(\lambda/\mu+2)} - bN(b)\right) + \tfrac{10}{3}c = 0,$$

$$(D)\quad \left[2+\left(\frac{\lambda}{\mu}+2\right)b\right](A_0+A_2)+4A_2-(6B_2-5A_2)$$
$$\times\left\{\frac{\lambda}{\mu}-\frac{b}{24}\left[\left(3\frac{\lambda}{\mu}+2+\left(\frac{\lambda}{\mu}+2\right)b\right)L(b)+6\frac{\lambda}{\mu}M(b)+2\left(\frac{\lambda}{\mu}+2\right)L'(b)\right]\right\}=0,$$

$$(E)\quad 3(A_0+A_2)+5A_2+(6B_2-5A_2)\left[1-\frac{b}{8}[L(b)-2M'(b)]\right]=0.$$

The functions represented by the infinite series may be otherwise expressed, depending ultimately upon the exponential integral

$$\int_0^x e^{\frac{1}{4}bx^2}dx.$$

For the purposes of computation, however, it is doubtful whether they can be reduced to expressions more convenient than the series, which converge with satisfactory rapidity unless b is relatively large. In applying the results to a body comparable in size and mass with the earth, for example, it can hardly be supposed that b is as great as 2; and for $b=2$ it is found that eight or ten terms of the series suffice, even in case of the function L', which is the least rapidly convergent of all.

§ 6. *Modified procedure for solving equations.*

It may be well to point out that by changing the procedure the algebraic work involved in the solution of equations (25) and (26) can be somewhat abridged.

The equations of equilibrium of a volume element may be abbreviated by introducing the elementary rotations ϖ_r, ϖ_θ, ϖ_ϕ. The general forms of these equations are given in Part IV of this paper, equations (117), $\cdots$, (123). When the strain is symmetrical about the initial line of the system of coordinates, ϖ_r and ϖ_θ vanish, and the equations reduce to the two following, which are identical with (1) and (2);

$$(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r\sin\theta}\frac{\partial}{\partial\theta}(\varpi_\theta\sin\theta)+\rho F_r=0,$$

$$(\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}+\frac{2\mu}{r\sin\theta}\frac{\partial}{\partial r}(r\varpi_\phi\sin\theta)+\rho F_\theta=0.$$

If we now write

$$\Delta=yP_2,\qquad 2\varpi_\phi=z\frac{\partial P_2}{\partial\theta},$$

and introduce everywhere y and z instead of e and α, equations (25) and (26) become

$$(\lambda+2\mu)x\frac{dy}{dx}+6\mu z + \rho g a\left[\frac{9}{5x^3}\int_0^x y'x'^4\,dx' - \frac{6x^2}{5}\int_x^1 \frac{y'dx'}{x'} + yx^2 - (\tfrac{4}{5}e_1+\tfrac{2}{3}c)x^2\right]=0,$$

$$(\lambda+2\mu)y+\mu\frac{d(xz)}{dx} - \rho g a\left[\frac{3}{5x^3}\int_0^x y'x'^4dx' + \frac{3x^2}{5}\int_x^1 \frac{y'dx'}{x'} + (\tfrac{2}{5}e_1+\tfrac{1}{3}c)x^2\right]=0.$$

It is easily shown that these are satisfied by series of the following forms:

$$y=C_2x^2+C_4x^4+\cdots, \qquad z=D_2x^2+D_4x^4+\cdots,$$

in which the coefficients satisfy the equations

$$\frac{C_{m+2}}{C_m}=-\frac{m+6}{m(m+5)}\cdot\frac{\rho g a}{\lambda+2\mu},$$

$$\frac{D_{m+2}}{C_m}=\frac{1}{m(m+5)}\cdot\frac{\rho g a}{\mu},$$

and an additional relation obtained by equating to 0 the coefficients of x^2 after substituting the values of y and z in the differential equations. One coefficient thus remains arbitrary.*

It is now possible to determine e and α from the equations

$$4\alpha-2e-\tfrac{2}{3}x\frac{de}{dx}=y, \qquad \tfrac{2}{3}\left(e-2\alpha-x\frac{d\alpha}{dx}\right)=z.$$

Thus if we assume

$$e=\Sigma A_m x^m, \qquad \alpha=\Sigma B_m x^m,$$

we find

$$(m+3)A_m-6B_m=-\tfrac{3}{2}C_m,$$

$$A_m-(m+2)B_m=\tfrac{3}{2}D_m,$$

from which A_m and B_m are determined in terms of C_m and D_m (and therefore in terms of the one arbitrary constant above mentioned), except in the case † $m=0$. In this case we may take A_0 arbitrarily and make

$$B_0=\tfrac{1}{2}A_0.$$

* Another solution consists of ascending series beginning with $C_{-3}x^{-3}$ and $D_{-3}x^{-3}$; but negative powers of x are not permissible in the physical problem.

† The case $m=-5$ is excluded by physical considerations.

That is, the values of e and α contain the same powers of x as the values of y and z, together with the absolute terms A_0 and $\frac{1}{2}A_0$. Two arbitrary constants remain for satisfying the surface conditions.

Although this process results in some saving of algebraic work, it does not appear to suggest any shortening of the labor involved in numerical computations such as those given below.

It is of some interest to note the form of the differential equation resulting from the elimination of z:

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-6y+\frac{\rho ga}{\lambda+2\mu}\left(x^3\frac{dy}{dx}+6x^2y\right)=0.$$

We obtain also the equation

$$x^2\frac{d^2z}{dx^2}+2x\frac{dz}{dx}-6z-\frac{\rho ga}{\mu}x^2y=0;$$

but the complete elimination of y obviously leads to an equation of the fourth order in z.

§ 7. *External potential and inertia-constant.*

One of the quantities which indicates the yielding of the body is the change in the external potential, or (what amounts to the same thing) in the difference between the principal moments of inertia.

Let I denote the moment of inertia about a diameter before strain, C and A the principal moments after strain (the former being taken about the axis of symmetry), and let

$$f=\frac{C-A}{I}. \tag{73}$$

The value contributed to $C-A$ by an elementary shell of thickness dr and density

$$-\rho\Delta=-\rho\left(4\alpha-2e-\tfrac{2}{3}r\frac{de}{dr}\right)P_2$$

is

$$\tfrac{4}{5}\pi r^4\rho\left(4\alpha-2e-\tfrac{2}{3}r\frac{de}{dr}\right)dr,$$

while the part contributed by the surface inequality is

$$\tfrac{8}{15}\pi\rho a^5 e_1;$$

hence

$$C-A=\frac{4\pi\rho}{5}\int_0^a\left(4\alpha-2e-\tfrac{2}{3}r\frac{de}{dr}\right)r^4dr+\tfrac{8}{15}\pi\rho a^5e_1. \tag{74}$$

Integrating by parts the term containing de/dr, we obtain

$$(75)\qquad C-A=\frac{16\pi\rho}{15}\int_0^a(3\alpha+e)r^4dr=\frac{16\pi\rho a^5}{15}\int_0^1(3\alpha+e)x^4dx,$$

and

$$(76)\qquad f=\frac{C-A}{I}=2\int_0^1(3\alpha+e)x^4dx.$$

This accords with the result already given for the potential due to the change of configuration. Thus the external potential is known to have the value

$$-\frac{\gamma(C-A)}{r^3}P_2,$$

while it is also represented by the first term of the value of V given in (20) if the upper limit of integration be a; i. e., we must have

$$-\frac{\gamma(C-A)}{r^3}P_2=-\frac{4gP_2}{5ar^3}\int_0^r(3\alpha'+e')r'^4dr'=-\frac{16\pi\rho\gamma P_2}{15r^3}\int_0^a(3\alpha+e)r^4dr,$$

which agrees with the result given above.

Using the values above found for e and α, we may write

$$(77)\quad f=\frac{C-A}{I}=A_0+A_2+(6B_2-5A_2)\left[\frac{1}{7}+\frac{b}{12}\left(2G(b)-3\frac{\lambda}{\mu}F(b)\right)\right],$$

in which

$$(78)\qquad F(b)=\sum_0^\infty(-1)^n\frac{n+3}{7\cdot9\cdots(2n+9)}b^n,$$

$$(79)\qquad G(b)=\sum_0^\infty(-1)^n\frac{(n+1)(n+3)}{7\cdot9\cdots(2n+9)}b^n.$$

§8. *Case of incompressibility.*

In the case of incompressibility $\lambda=\infty$, $b=0$. Putting $b'=ga\rho/\mu$, we find

$$(80)\qquad \begin{aligned} bL(b)&=\frac{ga\rho}{\lambda+2\mu}\left(-6\frac{\lambda}{\mu}\right)\frac{3}{7\cdot9}=-\tfrac{2}{7}b',\\ bM(b)&=\frac{ga\rho}{\lambda+2\mu}\left(7\frac{\lambda}{\mu}\right)\frac{3}{7\cdot9}=\tfrac{1}{3}b',\\ bN(b)&=\frac{ga\rho}{\lambda+2\mu}\left(\frac{\lambda}{\mu}\right)\frac{3}{7\cdot9}=\tfrac{1}{21}b',\\ bL'(b)&=\frac{ga\rho}{\lambda+2\mu}\left(-6\frac{\lambda}{\mu}\right)\frac{6}{7\cdot9}=-\tfrac{4}{7}b',\\ bM'(b)&=\frac{ga\rho}{\lambda+2\mu}\left(7\frac{\lambda}{\mu}\right)\frac{6}{7\cdot9}=\tfrac{2}{3}b'. \end{aligned}$$

Equation (C) now requires that

$$6B_2 - 5A_2 = 0, \tag{81}$$

$$(6B_2 - 5A_2)\frac{\lambda}{\mu} = \frac{3b'}{20}\left[4A_0 + 4\left(1 + \frac{35}{3b'}\right)A_2 + \tfrac{10}{3}c\right]. \tag{82}$$

while (D) and (E) give

$$(6B_2 - 5A_2)\frac{\lambda}{\mu} = (2 + b')A_0 + (6 + b')A_2, \tag{83}$$

$$3A_0 + 8A_2 = 0. \tag{84}$$

Solving, we obtain

$$A_0 = \frac{8b'}{38 + 4b'}c, \qquad A_2 = -\frac{3b'}{38 + 4b'}c, \qquad B_2 = -\frac{5b'}{2(38 + 4b')}c, \tag{85}$$

while all other coefficients in the values of e and α vanish. We thus have

$$\frac{e}{c} = \frac{b'}{38 + 4b'}(8 - 3x^2), \tag{86}$$

$$\frac{\alpha}{c} = \frac{b'}{2(38 + 4b')}(8 - 5x^2), \tag{87}$$

which agree with the known solution for the case of incompressibility.

We find also

$$f = A_0 + A_2 = \frac{5b'c}{38 + 4b'}, \tag{88}$$

which is identical with the surface value of e, as should be the case when $\lambda = \infty$.

§ 9. *Case of zero rigidity.*

This is the only case in which the solution fails. If $\mu = 0$, equations (25) and (26) cannot be satisfied unless $\Delta = 0$; and if $\Delta = 0$, the surface condition (32) cannot be satisfied unless $\lambda = \infty$.

§ 10. *Numerical results.*

In applying the formulas above found, it is convenient to choose a definite value of λ/μ and make computations for a series of values of b or $g\rho a/(\lambda + 2\mu)$. Some of the results thus found are given in the accompanying tables and diagrams,* which for the most part are self-explanatory. The quantities of particular interest are the surface values of e and α and f. As an aid to the

* In all the diagrams it is to be understood that ordinates are multiplied by c.

verification of the computations, the values of the functions L, M, etc., are included in the tables.

TABLE I. *Results for the case* $\lambda/\mu = 1$.

(All numbers below the horizontal line are to be multiplied by e. Numbers in square brackets [ ] obtained by interpolation; all others computed.)

$b = \frac{ga\rho}{\lambda + 2\mu}$	2	1.8	1.6	1.4	1.2	1	.8	.667
$\mu/ga\rho$	.167	.185	.208	.238	.278	.333	.417	.500
$L(b)$	.8946	.9347	.9773	1.0224	1.0703	1.1212	1.1753	1.2133
$M(b)$	.4618	.4746	.4879	.5018	.5163	.5315	.5474	.5585
$N(b)$	.2579	.2641	.2706	.2774	.2844	.2917	.2994	.3026
$L'(b)$	1.4000	1.4978	1.6042	1.7194	1.8444	1.9801	2.1273	2.2326
$M'(b)$	.7987	.8319	.8668	.9039	.9434	.9853	1.0298	1.0610
$F(b)$	.03798	.03880	.03964	.04052	.04142	.04236	.04333	.04400
$G(b)$	.02999	[.031]	[.033]	[.034]	[.036]	.03758	.03936	.04061
$6B_2 - 5A_2$	3.710	2.857	2.141	1.552	1.067	.7004	.4152	.2701
$A_0 = e_0$	.5146	.5251	.5218	.5058	.4758	.4327	.3760	.3303
A_2	—.5752	—.4954	—.4228	—.3570	—.2964	—.2405	—.1883	—.1551
B_2	.139	.0633	.0045	—.0388	—.0677	—.0837	—.0877	—.0842
e_1	.216	.230	.238	.241	.237	.225	.204	.1843
f	.436	.415	.390	.361	.327	.289	.246	.213
α_1	.253	.224	.196	.169	.142	.117	.0927	.0767

TABLE II. *Results for the case* $\lambda/\mu = 2$.

(Numbers below the horizontal line to be multiplied by e. Numbers in square brackets obtained by interpolation; all others computed.)

$b = \frac{ga\rho}{\lambda + 2\mu}$	2	1.8	1.6	1.4	1.2	1	.8	.667
$\mu/ga\rho$	.125	.139	.156	.179	.208	.250	.3125	.375
$L(b)$	.6667	.7019	.7394	.7793	.8218	.8671	.9153	.9493
$M(b)$	.7117	.7312	.7516	.7729	.7952	.8185	.8428	.8597
$N(b)$	.2959	.3029	.3103	.3179	.3258	.3341	.3427	.3467
$L'(b)$	.9922	1.0771	1.1697	1.2706	1.3806	1.5005	1.6311	1.7367
$M'(b)$	1.2321	1.2829	1.3365	1.3933	1.4537	1.5179	1.5860	1.6328
$F(b)$	.03798	[.0388]	[.0396]	[.0405]	[.0414]	.04236	.04333	.04400
$G(b)$	.02999	[.031]	[.033]	[.034]	[.036]	.03758	.03936	.04061
$6B_2 - 5A_2$	3.749	2.915	2.209	1.623	1.142	.7577	.4596	.3056
$A_0 = e_0$	.5537	.5852	.5984	.5935	.5698	.528	.4666	.4144
A_2	—.4655	—.4309	—.3938	—.3542	—.3118	—.267	—.2195	—.1862
B_2	.2369	.1267	.0400	—.0247	—.0695	—.0962	—.1063	—.1042
e_1	.2964	.3078	.3135	.3131	.3049	.289	.2611	.236
f	.519	.496	.469	.438	.401	.358	.307	.269
α_1	.2914	.259	.228	.199	.170	.142	.1140	.096

TABLE III. *Results for the case* $\lambda/\mu = 3$.

(Numbers below the horizontal line to be multiplied by c. Numbers in square brackets obtained by interpolation; all others computed.)

$b = \frac{g a \rho}{\lambda + 2\mu}$	2	1.8	1.6	1.4	1.2	1	.8	.667
$\mu/g a\rho$	.10	.111	.125	.143	.167	.200	.250	.300
$L(b)$	.4388	.4691	.5016	.5362	.5733	.6129	.6553	.6853
$M(b)$	.9616	.9880	1.0154	1.0441	1.0740	1.1054	1.1381	1.1610
$N(b)$	.3329	.3417	.3499	.3584	.3672	.3764	.3861	.3906
$L'(b)$	.5840	.6566	.7354	.8218	.9168	1.0212	1.1349	1.2414
$M'(b)$	1.6654	1.7339	1.8060	1.8826	1.9641	2.0505	2.1423	2.2046
$F(b)$	.03798	.03880	.03964	.04052	.04142	.04236	.04333	.04400
$G(b)$	.02999	[.031]	[.033]	[.034]	[.036]	.03758	.03936	.04061
$6B_2 - 5A_2$	3.910	3.032	2.320	1.721	1.224	.823	.497	.343
$A_0 = e_0$	.545	.598	.643	.654	.642	.605	.532	.489
A_2	−.340	−.348	−.351	−.339	−.317	−.285	−.239	−.214
B_2	.368	.215	.094	.0043	−.0602	−.1003	−.1163	−.1215
e_1	.3487	[.365]	.370	.369	.360	.341	[.314]	.281
f	.580	.556	.533	.501	.463	.417	[.370]	.318
α_1	.327	.290	.258	.226	.195	.164	.131	.112

In Figs. 2 and 3 are shown curves representing e_1, α_1 and f as functions of $\mu/g\rho a$, for certain constant values of λ/μ. It is seen that, as λ/μ increases, the curves for e_1 and f both approach the curve (A) which represents the case $\lambda = \infty$. For a given value of μ both e_1 and f increase with λ; in other words, the effect of compressibility is to decrease both the ellipticity and the external potential corresponding to a certain rigidity. This effect is much more important upon e_1 than upon f. Thus if $\mu = .35\rho g a$, a variation of λ/μ from 1 to ∞ would change f/c only from .279 to .289, while e_1/c would range from .221 to .289. For smaller values of μ the effect of compressibility is greater. An interesting feature is the fact that for a given value of λ/μ there is a maximum value of e_1; thus if $\lambda = \mu$, a decrease of μ below about $.24\rho g a$ causes e_1 to decrease.

Fig. 3 shows also that the value of α_1 is increased to an important extent by compressibility.

II. STRAIN OF AN ELASTIC SPHERE COVERED BY A SHALLOW OCEAN.

§ 1. *Statement of problem.*

In applying the theory of the deformation of a sphere to the case of the earth, an important question is the effect of the ocean. Some light on the order of magnitude of this effect is thrown by the solution of the following problem:

To determine the strain of a gravitating elastic sphere, the surface of which is covered by a shallow ocean in equilibrium.

The solution above given may without difficulty be extended to this case, as follows.

§ 2. *Forces due to ocean inequality.*

Let $c'\rho$ = density of ocean,

$\mathfrak{e}$ = ellipticity of ocean surface.

Then there must be introduced into the solution the forces due to an ocean layer of thickness

$$-\tfrac{2}{3}(\mathfrak{e}-e_1)aP_2.$$

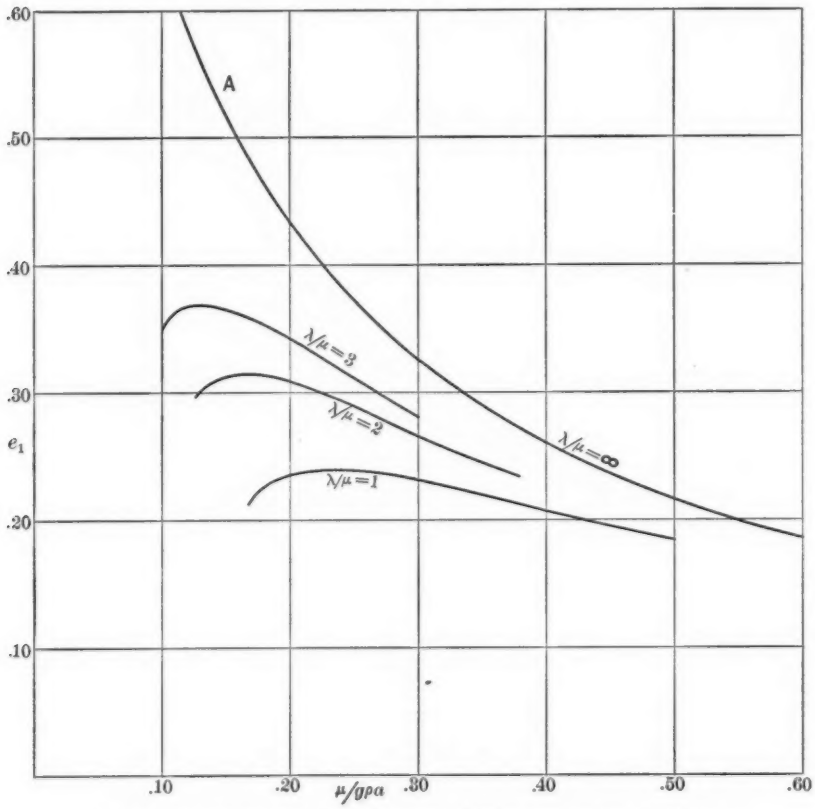

FIG. 2. Curves showing effect of compressibility upon surface ellipticity.

These forces will be (a) the bodily force due to the attraction of the layer, and (b) the surface pressure due to the weight of the layer.

(a) The attraction is given by the potential

$$(89) \qquad V' = -\tfrac{8}{15}\pi\gamma c'\rho(\mathfrak{e}-e_1)r^2P_2 = -\frac{2g}{5a}c'(\mathfrak{e}-e_1)r^2P_2,$$

which adds to (23) and (24) the terms

$$\rho \frac{\partial V'}{\partial r} = -\frac{4g\rho}{5a} c'(\mathfrak{e} - e_1) r P_2,$$

$$\frac{\rho}{r} \frac{\partial V'}{\partial \theta} = \frac{3g\rho}{5a} c'(\mathfrak{e} - e_1) r \sin 2\theta,$$

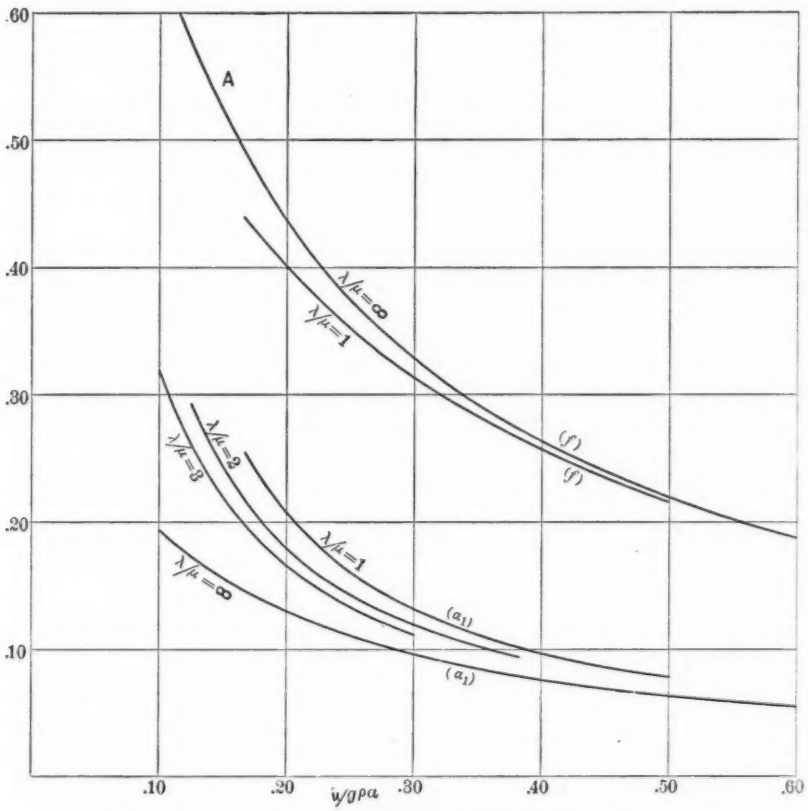

FIG. 3. Curves showing effect of compressibility upon angular displacement and external potential.

and therefore to (25) and (26) the terms

(90) $$-\tfrac{4}{5} g\rho a c'(\mathfrak{e} - e_1) x^2,$$

(91) $$\tfrac{3}{5} g\rho a c'(\mathfrak{e} - e_1) x^2.$$

(*b*) The weight of the ocean inequality per unit area is

$$-\tfrac{3}{5}c'\rho ga(\mathfrak{e}-e_1)P_2,$$

hence there must be added to the second member of (32) the term

$$\tfrac{3}{5}\rho gac'(\mathfrak{e}-e_1). \tag{92}$$

§ 3. *Value of ellipticity of ocean surface.*

The value of $\mathfrak{e}$ (the ellipticity of a surface which is level under the action of gravity and the disturbing forces) may be expressed in terms of f, e_1 and c, as follows:

For the solid sphere the value of $C-A$ is If, while for the ocean inequality it is $Ic'(\mathfrak{e}-e_1)$. Hence by a familiar result in the theory of potential,

$$I[f+c'(\mathfrak{e}-e_1)]=\tfrac{3}{5}Ma^2(\mathfrak{e}-\tfrac{1}{2}c). \tag{93}$$

Or, since for a sphere of uniform density $I=\frac{2}{5}Ma^2$, we have

$$f+c'(\mathfrak{e}-e_1)=\tfrac{5}{3}(\mathfrak{e}-\tfrac{1}{2}c), \tag{94}$$

whence

$$\mathfrak{e}=\frac{6f-6c'e_1+5c}{10-6c'}, \tag{95}$$

and

$$\mathfrak{e}-e_1=\frac{6f-10e_1+5c}{10-6c'}. \tag{96}$$

The terms to be added to (25), (26) and (32) thus become

$$\rho ga\left[-\tfrac{4}{5}c'\frac{6f-10e_1+5c}{10-6c'}x^2\right], \tag{97}$$

$$\rho ga\left[\tfrac{3}{5}c'\frac{6f-10e_1+5c}{10-6c'}x^2\right], \tag{98}$$

$$\rho ga\left[\tfrac{2}{3}c'\frac{6f-10e_1+5c}{10-6c'}\right]. \tag{99}$$

§ 4. *Statement of solution.*

The values of e and α for this case are expressed by the same equations (A) and (B) as before, but two of the three equations (C), (D) and (E) which determine the constants must be changed as follows.

In (C) replace $\frac{10}{3}c$ by $[50c+6c'(6f-10e_1)]/[3(5-3c')]$.

Add to the first member of (D):

$$\left(\frac{\lambda}{\mu}+2\right)b\,\frac{c'(5c+6f-10e_1)}{10-6c'}.$$

Equation (E) remains unchanged.

Equations (C) and (D) are thus replaced by the following:

$$(C_1)\qquad \begin{aligned} &4(A_0+A_2)+\frac{140}{3b(\lambda/\mu+2)}A_2 \\ &\quad+(6B_2-5A_2)\left(2-\frac{20\lambda/\mu}{3b(\lambda/\mu+2)}-bN(b)\right) \\ &\quad+\frac{12c'}{5-3c'}f-\frac{20c'}{5-3c'}e_1+\frac{50}{3(5-3c')}c=0, \end{aligned}$$

$$(D_1)\qquad \begin{aligned} &\left[2+\left(\frac{\lambda}{\mu}+2\right)b\right]A_0+\left[6+\left(\frac{\lambda}{\mu}+2\right)b\right]A_2-(6B_2-5A_2) \\ &\times\left\{\frac{\lambda}{\mu}-\frac{b}{24}\left[\left(3\frac{\lambda}{\mu}+2+\left(\frac{\lambda}{\mu}+2\right)b\right)L(b)+6\frac{\lambda}{\mu}M(b)+2\left(\frac{\lambda}{\mu}+2\right)L'(b)\right]\right\} \\ &+\left(\frac{\lambda}{\mu}+2\right)b\left(\frac{3c'}{5-3c'}f-\frac{5c'}{5-3c'}e_1+\frac{5c'}{2(5-3c')}c\right)=0. \end{aligned}$$

When f and e_1 are replaced by their values

$$(100)\qquad e_1=A_0+A_2+2\tfrac{1}{4}(6B_2-5A_2)bL(b),$$

$$(101)\qquad f=A_0+A_2+(6B_2-5A_2)\left[\frac{1}{7}+\frac{b}{12}\left(2G(b)-3\frac{\lambda}{\mu}F(b)\right)\right],$$

the equations become

$$(C')\qquad \begin{aligned} &\frac{20(1-c')}{5-3c'}(A_0+A_2)+\frac{140}{3b(\lambda/\mu+2)}A_2+(6B_2-5A_2) \\ &\times\left[2-\frac{20\lambda/\mu}{3b(\lambda/\mu+2)}-bN(b)+\frac{4c'}{5-3c'}\left(\tfrac{3}{7}+\tfrac{1}{2}bG(b)-\frac{3\lambda}{4\mu}bF(b)-2\tfrac{5}{4}bL(b)\right)\right] \\ &+\frac{50}{3(5-3c')}c=0, \end{aligned}$$

$$(D')\qquad \begin{aligned} &\left[2+\left(\frac{\lambda}{\mu}+2\right)b\frac{5(1-c')}{5-3c'}\right](A_0+A_2)+4A_2-(6B_2-5A_2) \\ &\times\left\{\frac{\lambda}{\mu}-\frac{1}{24}\left[\left(3\frac{\lambda}{\mu}+2+\left(\frac{\lambda}{\mu}+2\right)b\right)bL(b)+6\frac{\lambda}{\mu}bM(b)+2\left(\frac{\lambda}{\mu}+2\right)bL'(b)\right]\right. \\ &\left.-\left(\frac{\lambda}{\mu}+2\right)b\frac{c'}{5-3c'}\left[\tfrac{3}{7}+\tfrac{1}{2}bG(b)-\frac{3\lambda}{4\mu}bF(b)-2\tfrac{5}{4}bL(b)\right]\right\} \\ &+\left(\frac{\lambda}{\mu}+2\right)b\frac{5c'}{2(5-3c')}c=0. \end{aligned}$$

To these we add equation (E) unchanged:

$$(E')\quad 3(A_0+A_2)+5A_2+(6B_2-5A_2)\left[1+\frac{b}{8}\left(L(b)-2M'(b)\right)\right]=0.$$

These three equations determine the constants A_0, A_2, B_2.

§ 5. *Value of inertia-constant for combined sphere and ocean.*

Let f' represent the value of $(C-A)/I$ for the whole system, then

$$f'=f+c'(e-e_1),$$

which by (96) reduces to

$$f'=\frac{5}{5-3c'}f-\frac{5c'}{5-3c'}e_1+\frac{5c'}{2(5-3c')}c. \tag{102}$$

§ 6. *Case of incompressibility.*

If $\lambda=\infty$, equations (A) and (B) hold, and equations (C') and (D') require that

$$6B_2-5A_2=0. \tag{103}$$

Also, on writing

$$\frac{\lambda}{\mu}(6B_2-5A_2)=C, \tag{104}$$

(C'), (D') and (E') become

$$C-7A_2-\frac{3(1-c')}{5-3c'}b'(A_0+A_2)-\frac{5b'}{2(5-3c')}c=0, \tag{105}$$

$$C-4A_2-\left(2+\frac{5(1-c')}{5-3c'}b'\right)(A_0+A_2)-\frac{5c'b'}{2(5-3c')}c=0, \tag{106}$$

$$3(A_0+A_2)+5A_2=0. \tag{107}$$

The solution of these equations gives

$$\begin{aligned} A_0&=\frac{20(1-c')b'}{19(5-3c')+10(1-c')b'}c,\\ A_2&=-\frac{15(1-c')b'}{38(5-3c')+20(1-c')b'}c,\\ B_2&=-\frac{25(1-c')b'}{76(5-3c')+40(1-c')b'}c. \end{aligned} \tag{108}$$

If $c'=0$, these reduce to (85), as of course they should.

§ 7. *Numerical results.*

For the purpose of showing the order of importance of the ocean effect, computations have been made with a value of c' equal to the ratio of the density of sea-water to the mean density of the earth. This value is taken as $5/27$.

The computations made cover the two cases $\lambda/\mu = 1$, $\lambda/\mu = \infty$, and the results are shown in the accompanying table and diagrams.

For the case of incompressibility

$$(109)\quad A_0 = \frac{22b'}{114 + 11b'}c = \frac{22}{114\,\dfrac{\mu}{g\rho a} + 11}\,c, \qquad A_2 = -\tfrac{3}{8}A_0, \qquad B_2 = -\tfrac{5}{16}A_0,$$

$$(110)\quad \frac{e}{c} = \frac{22}{114\,\dfrac{\mu}{g\rho a} + 11}(1 - \tfrac{3}{8}x^2),$$

$$(111)\quad \frac{\alpha}{c} = \frac{11}{114\,\dfrac{\mu}{g\rho a} + 11}(1 - \tfrac{5}{8}x^2),$$

$$(112)\quad \frac{e_1}{c} = \frac{f}{c} = \frac{1}{c}(A_0 + A_2) = \frac{5}{\dfrac{456}{11}\,\dfrac{\mu}{g\rho a} + 4}.$$

The last expression is written in a form for comparison with the value without ocean effect:

$$(113)\quad \frac{e_1}{c} = \frac{f}{c} = \frac{5}{38\,\dfrac{\mu}{g\rho a} + 4}.$$

From (102),

$$(114)\quad f' = \tfrac{11}{12}e_1 + \tfrac{5}{48}c.$$

Values of e_1, f and f' for this case, also corresponding values without ocean effect, are given in Table IV, while the same results are shown graphically in Fig. 4.

For $\lambda = \mu$, computations have been made by means of equations (A'), (B'), (C'), (D'), (E'), with the results given in Table V and shown graphically in Fig. 5.

Inspection of the curves shows that the effect of the ocean on the strain of the sphere is quite appreciable even in the case of incompressibility, and is very considerable in the case $\lambda = \mu$. A fact of equal importance in the interpretation of observational results as to the yielding of the earth is shown by the curves representing f'. These are nearly identical for the two cases $\lambda = \infty$, $\lambda = \mu$ (Figs. 4 and 5), and in both cases are very considerably above the curves representing f. It thus appears that an important part of the change of external potential is due to the ocean—a result which will be referred to again in the following discussion of the yielding of the earth.

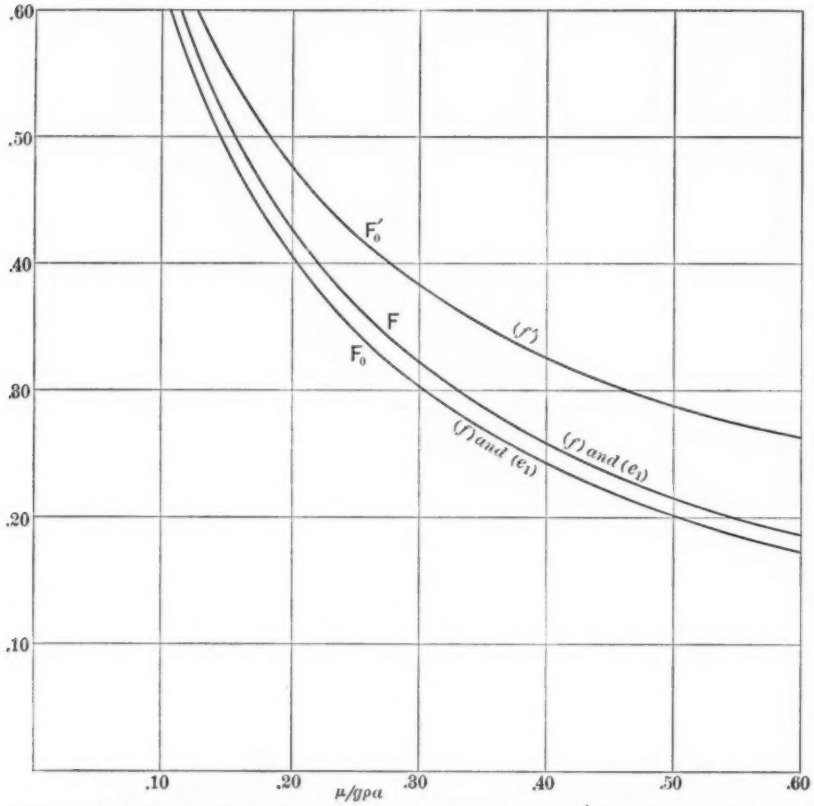

FIG. 4. Curves showing effect of ocean in case $\lambda = \infty$. (F_0 and F_0' with ocean, F without.)

TABLE IV. *Effect of ocean in case* $\lambda = \infty$.

$\frac{\mu}{g\rho a}$	Without ocean.		With ocean.		
	$\frac{a_1}{c}$	$\frac{e_1}{c} = \frac{f}{c}$	$\frac{a_1}{c}$	$\frac{e_1}{c} = \frac{f}{c}$	$\frac{f'}{c}$
0	.375	1.250	.375	1.250	1.250
.05	.254	.848	.247	.823	.858
.10	.193	.642	.184	.613	.666
.15	.155	.516	.146	.488	.551
.20	.129	.431	.122	.407	.477
.25	.111	.370	.104	.348	.423
.30	.097	.325	.091	.304	.383
.35	.087	.289	.081	.270	.352
.40	.078	.260	.073	.243	.327
.45	.071	.237	.066	.221	.307
.50	.065	.217	.061	.202	.289
.55	.060	.201	.056	.187	.275
.60	.056	.187	.052	.173	.263

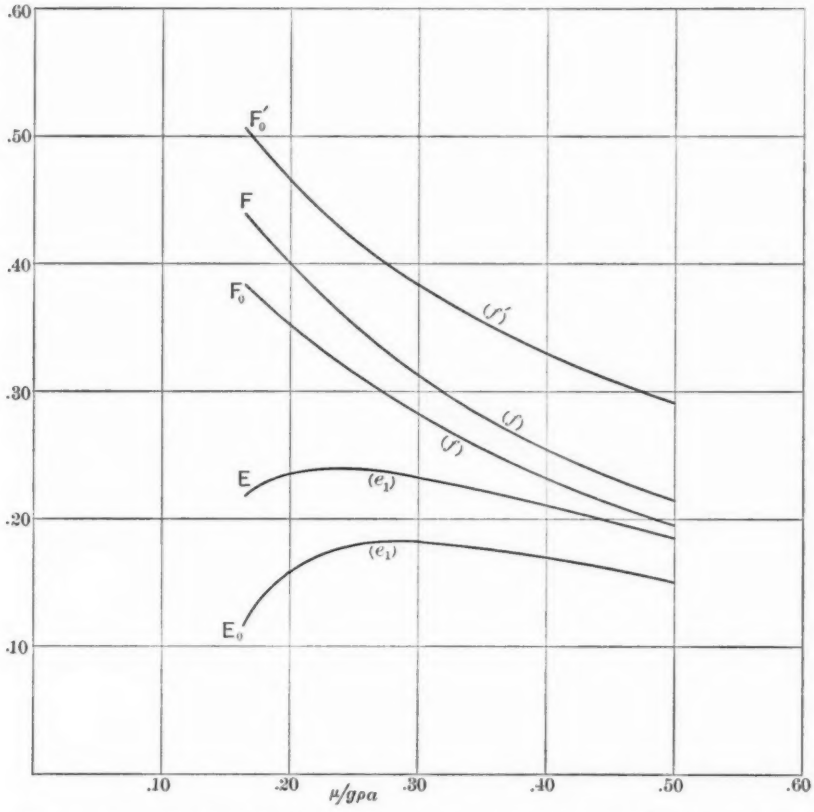

FIG. 5. Curves showing effect of ocean in case $\lambda = \mu$. (Curves E, F without ocean; E_0, F_0, F_0' with ocean.)

TABLE V. *Effect of ocean in case* $\lambda = \mu$.

b	$\frac{\mu}{g\rho a}$	Without ocean.			With ocean.			
		$\frac{a_1}{e}$	$\frac{e_1}{e}$	$\frac{f}{e}$	$\frac{a_1}{e}$	$\frac{e_1}{e}$	$\frac{f}{e}$	$\frac{f'}{e}$
2	.167	.253	.216	.436	.257	.119	.379	.505
1.8	.185	.224	.230	.415	.229	.145	.365	.485
1.6	.208	.196	.238	.390	.198	.163	.345	.458
1.4	.238	.169	.241	.361	.170	.176	.322	.429
1.2	.278	.142	.237	.327	.143	.181	.294	.397
1.0	.333	.117	.225	.289	.118	.179	.263	.363
.8	.417	.093	.204	.246	.094	.167	.225	.322
.667	.500	.077	.184	.213	.077	.151	.194	.291

III. Effect of compressibility upon estimates of the rigidity of the earth.

The estimates of the rigidity of the earth, made originally by Lord Kelvin and since by others, are based upon a comparison of what is known regarding the actual yielding of the earth under known disturbing forces with the computed yielding of an ideal body under like forces. This ideal body is a homogeneous, elastic, incompressible sphere of mass and size equal to those of the earth. It is, of course, not to be expected that such a comparison can lead to anything more than a rough average value of the modulus of rigidity of the material of the earth, since the actual earth is certainly far from uniform in density, and probably far from uniform in elastic properties. It is, however, a matter of some interest to notice in what way these estimates are modified when compressibility is taken into account. In answering this question it is needful to refer briefly to the present status of the evidence regarding the actual yielding of the earth.*

§ 1. *Nature of evidence of yielding of the earth.*

Evidence regarding the actual yielding of the earth relates to the two quantities above represented by e_1 and f. The former is the surface ellipticity due to strain, the latter measures the change in the gravitational potential at the earth's surface due to the strain. The number f is, in fact, the value of $(C - A)/I$ due to the strain. It may be remarked that for a homogeneous *incompressible* body e_1 and f are equal, but for a compressible body f depends upon the values of both e and α throughout the body.

There are three lines of evidence of yielding of the earth to tidal or centrifugal forces: (*a*) Tidal observations, (*b*) experiments with the horizontal pendulum, (*c*) the movement of the axis of rotation within the earth as determined by latitude observations.

(*a*) *Nature of evidence from tidal observations.* The height of an equilibrium-tide at any locality can be computed on the assumption that the solid earth is unyielding, also on the assumption that f and e_1 have any definite values. A comparison of these results with the observed tidal oscillation gives evidence as to the actual values of f and e_1. It may, in fact, be shown without difficulty that if the attraction of the ocean itself be neglected, the ratio of the actual fluctuation of the water relative to the solid ground to the fluctuation on the assumption of no yielding of the solid earth is

* A clear and non-technical summary of this evidence is given by G. H. Darwin in a paper entitled "*The Rigidity of the Earth*," in Rivista di Scienza, vol. 5 (1909). See also a paper by A. E. H. Love entitled "*The Yielding of the Earth to Disturbing Forces*," in the Proceedings of the Royal Society, Series A, Vol. 84.

$$\frac{c+f-2e_1}{c}=1+\frac{f}{c}-2\frac{e_1}{c}.$$

It is here assumed that the tide has its equilibrium value, an assumption which can reasonably be made only in the case of tides of long period. It is therefore only the fortnightly and monthly tides that have been available for this comparison.

It should be said also that for full generality the term f/c must be multiplied by $3I/Ma^2$. For a homogeneous sphere the value of this is $6/5$, but for the earth it is very nearly 1. It is the actual value for the earth that must be used in the application made below in § 2.

(*b*) *Nature of the evidence from experiments with the horizontal pendulum.* If the change in the direction of apparent gravity relative to the earth at any locality due to tidal forces can be measured, a comparison of this effect with the effect computed on the assumption of absolute rigidity of the earth will give evidence as to the actual yielding of the earth. The actual effect is so minute that it must be greatly magnified in order to become measurable. This magnification has been accomplished by means of the horizontal pendulum. It is not difficult to show that the actual deflection of such a pendulum due to tidal forces bears the same ratio to the deflection that would occur if the earth did not yield as the amplitude of an equilibrium tide bears to the amplitude it would have if the earth were unyielding. The quantity determined from horizontal pendulum observations is therefore identical with that inferred from tidal observations.

(*c*) *Nature of the evidence from the periodical variation of latitudes.* The periodic motion of the axis of rotation within the earth furnishes evidence as to the yielding of the earth to centrifugal forces. This evidence is of a different character from that obtained from the tides or the horizontal pendulum, in that it leads to an estimate of the value of f/c instead of to a relation between f/c and e_1/c. The change in the value of $(C-A)/I$ due to the yielding of the earth to given centrifugal forces may be inferred directly from a comparison of the actual period of the free nutation of the earth's axis with the period computed for an unyielding body having the earth's actual figure. It may, in fact, be shown that the actual nutation period is to a close approximation the same as that of an unyielding body having the figure which the earth would assume if centrifugal forces were annulled.*

*This was shown by S. S. HOUGH (Philosophical Transactions of the Royal Society, ser. *A*, 1896) for the case of a homogeneous, incompressible elastic spheroid, and has been proved by J. LARMOR and others to hold without other restriction as to the character of the body than that its density and elastic properties have approximate spherical symmetry. A proof of this was given by the present writer in a paper presented to the American Mathematical Society, San Francisco Section, Dec. 20, 1902, of which only an abstract was published, Bulletin of this Society, Vol. 9 (1903), p. 299. The reasoning employed was identical with that published by LARMOR, Proceedings of the Royal Society, ser. *A*, vol. 82 (1909).

§ 2. *Numerical results expressing actual yielding of the earth.*

The actual value of $(C-A)/I$ for the earth is $1/306$, so that the period of free nutation if it were absolutely rigid would be 306 sidereal days. The period of the movement of the axis about its mean position, as inferred from the observations of the International Latitude Service carried on systematically since 1899, has varied from 427 days to 442 days. The former gives

$$f = \tfrac{1}{306} - \tfrac{1}{427} = .00093;$$

the latter,

$$f = \tfrac{1}{306} - \tfrac{1}{442} = .00100.$$

Since the ratio of equatorial centrifugal force to gravity is $c = 1/289$, the corresponding values of f/c are .269 and .289. The former of these values has hitherto usually been accepted. It is nearly equal to $4/15$.

The first analysis of tidal observations, with reference to the question of the yielding of the earth, was made by G. H. DARWIN, the data and results being given in *Thomson and Tait's Natural Philosophy*. From all the data then available which were regarded as trustworthy, the value found for the ratio of the actual tidal amplitude to that computed on the assumption of an absolutely rigid earth was $.676 \pm .076$; but a separate analysis of the more consistent observations gave $.931 \pm .056$. The result was not regarded by the author as conclusive.* Analyses since made by others have been confirmatory of the former of these values, and about the same result is inferred from observations with the horizontal pendulum.†

If this result be accepted, we have (very nearly)

$$1 + \frac{f}{c} - 2\frac{e_1}{c} = \frac{2}{3}. \tag{115}$$

And if in this we substitute

$$\frac{f}{c} = \frac{4}{15},$$

there results

$$\frac{e_1}{c} = \frac{3}{10}.$$

§ 3. *Comparison of actual yielding with that of an ideal elastic solid.*

It is pointed out by LOVE‡ that the above values of f/c and e_1/c cannot coexist in the case of a homogeneous, incompressible elastic solid having the earth's size and mass, and it is easily seen from the results found in the first

* "We thus see that the more consistent observations seem to bring out the tides more nearly to their theoretical equilibrium values with no elastic yielding of the solid." G. H. DARWIN, in *Thomson and Tait's Natural Philosophy*, Part II, p. 460 (edition of 1890).

† W. SCHWEYDAR, Beiträge zur Geophysik, vol. 9 (1908); O. HECKER, *Veröffentlichung des Kgl. Preussischen Geodätischen Institutes*, No. 32, Berlin (1907).

‡ Loc. cit.

part of this paper that the discrepancy is increased rather than diminished if the effect of compressibility is considered. This is shown clearly by Fig. 6, in which the curves represent the relation between e_1/c and f/c for several values of λ/μ, while the straight line AB represents equation (115), or

$$2\frac{e_1}{c} - \frac{f}{c} = \frac{1}{3}. \tag{116}$$

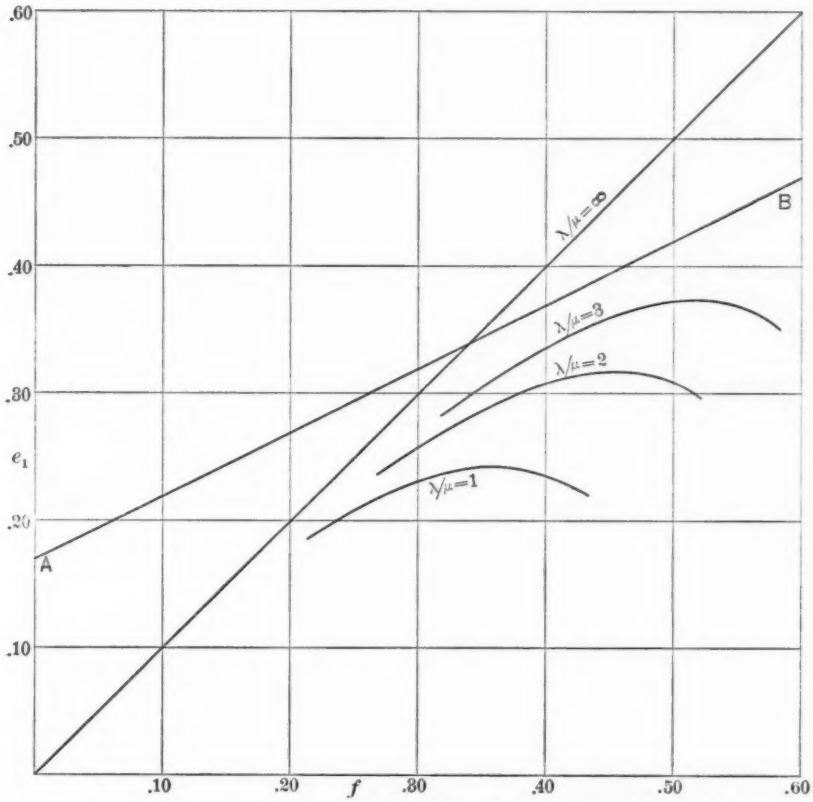

FIG. 6. Curves showing relation between e_1 and f for several values of λ/μ.

As λ/μ increases, the curves approach the limiting line $e_1 = f$; and until λ/μ reaches a value considerably greater than 3, the curves will fail to intersect AB. It is obvious that if λ/μ be taken great enough so that (116) can be satisfied, the value of f/c will be greater than $\frac{4}{15}$; being in fact $\frac{1}{3}$ for $\lambda = \infty$, and greater than $\frac{1}{3}$ for any finite value of λ/μ.

It may be remarked that the discrepancy between

$$\frac{f}{c}=\frac{1}{3}, \qquad \frac{e_1}{c}=\frac{1}{3},$$

and

$$\frac{f}{c}=\frac{4}{15}, \qquad \frac{e_1}{c}=\frac{3}{10},$$

is after all not remarkably great, considering the great delicacy of the observations upon which equation (116) is based. It may perhaps be still open to question whether the accepted values are so thoroughly reliable as to exclude the possibility of errors of the magnitude of these discrepancies.*

It may be noted, also, that a value of $1+f/c-2e_1/c$ much less than the greater of Darwin's results would remove the discrepancy. Thus the following values would be consistent for the ideal homogeneous sphere:

$\frac{\lambda}{\mu}$	$\frac{f}{c}$	$\frac{e_1}{c}$	$1+\frac{f}{c}-2\frac{e_1}{c}$
∞	.270	.270	.730
3	.270	.245	.780
2	.270	.235	.800
1	.270	.220	.830

Although the value of the surface ellipticity accepted as most reliable is more nearly consistent with the computed value for an incompressible than for a compressible sphere, it can hardly be supposed that this furnishes any important evidence as to the actual compressibility of the materials composing the earth. As stated above, only a roughly approximate agreement is to be expected between the behavior of the actual earth and that of an ideal body assumed to be uniform in density and elastic properties. Moreover, there are certain questions not generally considered which appear to require attention in any complete study of the theoretical basis of the comparison. One of these questions relates to the effect of the ocean.

* The most elaborate and apparently the most trustworthy series of observations heretofore made with the horizontal pendulum are those of HECKER (loc. cit.). A remarkable feature of the results obtained for the lunar effect upon the yielding of the earth is the variation of this effect with the azimuth of the moon. The value found for the ratio of the actual deflection of gravity to the deflection computed for a rigid earth is a minimum when the moon is on the meridian, and increases continually with the azimuth. The minimum value of the ratio is about 0.36, for azimuth 45° it is about 0.50, and for azimuth 67° about 0.90; and although as the azimuth approaches 90° the ratio becomes indeterminate, it appears to approach unity. No satisfactory explanation of this variation seems to have been suggested. If it is accepted as proving that the yielding of the earth varies greatly with the azimuth of the disturbing body, it appears to render of little value conclusions based upon the ordinary theory which assumes that the yielding is symmetrical about the axis of symmetry of the disturbing forces. At all events these results can hardly be regarded as establishing any definite value of the ratio of yielding. [The above numerical values of the ratio for different azimuths were obtained by measurement from the diagram given by HECKER showing the observed lunar effect and the effect computed for a rigid earth; assuming that the deflection of apparent gravity always agrees in azimuth with the moon.] HECKER's results are discussed by G. H. DARWIN in the paper already cited. This paper was received too late to permit here any quotation of suggestive comments.

§ 4. *Effect of the ocean on estimates of the yielding of the earth.*

The question as to the effect of the ocean involves two parts, — the first, what effect has the ocean on the actual yielding of the solid earth, and the second, what direct effect has the ocean on the observational results?

While no exact answer can be given to either of these questions, something can be inferred as to the degree of importance of the ocean effect from the results reached in Part II of this paper, and shown in Tables IV and V, and in Figs. 4 and 5. These results apply to the two cases $\lambda = \infty$ and $\lambda/\mu = 1$. It appears that the effect of the ocean upon the strain of the sphere is quite appreciable in both cases, and in the case $\lambda = \mu$ becomes of considerable importance. Although the angular displacement α_1 is affected but slightly, both e_1 and f are very considerably changed by the influence of the ocean.

Moreover, while these quantities are both materially decreased, the total external potential due to the solid sphere and ocean is considerably greater than the potential when the ocean is absent. This is true both for $\lambda = \mu$ and for $\lambda = \infty$, as is seen from the values of f''/c. (It is of interest to note that the curves showing the relation between f''/c and $\mu/g\rho a$ are nearly identical for the two cases.)

In estimating the value of μ from the value of $(C - A)/I$ inferred from the prolongation of the period of free nutation, it is therefore important to know whether the quantity determined applies to the solid earth alone or to the combined solid earth and ocean. Thus if it applies to the solid earth alone, the value $f/c = 0.27$ would correspond to

$$\frac{\mu}{g\rho a} = \text{about } 0.37,$$

while if 0.27 is the value of f''/c, we find

$$\frac{\mu}{g\rho a} = \text{about } 0.57.$$

The former would give $\mu = 1.30 \times 10^{12}$, while the latter would give $\mu = 2 \times 10^{12}$. These values are in C. G. S. units (dynes per square centimeter), the value of $g\rho a$ in these units being taken as 3.5×10^{12}.

Whether the change in the value of $(C - A)/I$ due to centrifugal forces, as inferred from the nutation period, applies to the solid earth alone, or whether an important part of it is due to the change in configuration of the ocean, is a question which must be answered in the light of the theory of the effect of yielding upon the nutation period. It seems a reasonable assumption that, for a disturbance for so long a period as 427 days, the ocean always has its equilibrium figure. If this be assumed in applying the equation of angular momen-

tum,* it is difficult to avoid the conclusion that the change in the value of $(C-A)/I$ inferred from the prolongation of the nutation period includes the yielding of the ocean; in other words, that the number $4/15$ is the value of f'/c rather than of f/c.

What effect the ocean may have upon observations with the horizontal pendulum is a more difficult question, because the equilibrium theory must be discarded in dealing with tides of short period.

IV. Strain of a compressible, gravitating elastic sphere due to disturbing forces of which the potential is a spherical harmonic of any degree not less than the second.

The following solution of this problem follows the general lines of the solution already given for the case in which the degree of the harmonic is 2. It in fact includes that as a special case. In comparing the algebraic results it must be noted that there is a slight change in the notation; the quantities e and α as used below must be multiplied by the factor $-\frac{3}{2}$ in order to agree with e and α in the solution given above.

§ 1. *Physical analysis of the problem.*

The physical analysis is identical with that already given in the special case, and need not be repeated.

§ 2. *Formation of the equations of equilibrium of an element of volume.*

(*a*) *General form of equations in polar coördinates.* The polar coördinates being r, θ, ϕ, let the corresponding displacements be

$$u_r,\ u_\theta,\ u_\phi,$$

the cubical expansion Δ, and the three components of rotation ϖ_r, ϖ_θ, ϖ_ϕ. Also let the components of total bodily force (per unit volume) be ρF_r, ρF_θ, ρF_ϕ. Then the equations of equilibrium for a volume element are the following:†

$$(117)\qquad (\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\varpi_\phi\sin\theta)-\frac{\partial\varpi_\theta}{\partial\phi}\right]=-\rho F_r,$$

$$(118)\qquad (\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}-\frac{2\mu}{r\sin\theta}\left[\frac{\partial\varpi_r}{\partial\phi}-\frac{\partial}{\partial r}(r\varpi_\phi\sin\theta)\right]=-\rho F_\theta,$$

$$(119)\qquad (\lambda+2\mu)\frac{1}{r\sin\theta}\frac{\partial\Delta}{\partial\phi}-\frac{2\mu}{r}\left[\frac{\partial}{\partial r}(r\varpi_\theta)-\frac{\partial\varpi_r}{\partial\theta}\right]=-\rho F_\phi;$$

* In the manner employed by Larmor in the paper cited above.

† See Love, *Treatise on the Mathematical Theory of Elasticity*, 2d edition (1906), p. 138, for the first members of these equations; p. 56, for the values of Δ, ϖ_r, ϖ_θ, ϖ_ϕ.

while the cubical expansion and the components of rotation are given by the following equations:

$$\Delta = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 u_r \sin\theta) + \frac{\partial}{\partial\theta}(r u_\theta \sin\theta) + \frac{\partial}{\partial\phi}(r u_\phi)\right], \tag{120}$$

$$2\varpi_r = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(r u_\phi \sin\theta) - \frac{\partial}{\partial\phi}(r u_\theta)\right], \tag{121}$$

$$2\varpi_\theta = \frac{1}{r \sin\theta}\left[\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(r u_\phi \sin\theta)\right], \tag{122}$$

$$2\varpi_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right]. \tag{123}$$

(*b*) *Assumptions.* The potential of the disturbing forces, being a spherical solid harmonic of degree i, is proportional to

$$S_i r^i,$$

in which S_i is a spherical surface harmonic, and therefore satisfies the partial differential equation

$$\frac{1}{\sin^2\theta}\frac{\partial^2 S_i}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_i}{\partial\theta}\right) + i(i+1)S_i = 0. \tag{124}$$

It will be assumed that

$$u_r = u S_i, \tag{125}$$

$$u_\theta = v\frac{\partial S_i}{\partial\theta}, \tag{126}$$

$$u_\phi = v\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi}, \tag{127}$$

in which u and v are independent of θ and ϕ. These assumptions give by virtue of (124)

$$\Delta = \left[\frac{1}{r^2}\frac{d(r^2 u)}{dr} - i(i+1)\frac{v}{r}\right] S_i, \tag{128}$$

$$2\varpi_r = 0, \tag{129}$$

$$2\varpi_\theta = \frac{1}{r}\left[u - \frac{d(rv)}{dr}\right]\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi}, \tag{130}$$

$$2\varpi_\phi = -\frac{1}{r}\left[u - \frac{d(rv)}{dr}\right]\frac{\partial S_i}{\partial\theta}, \tag{131}$$

and reduce (117), (118) and (119) to the following:

$$(132)\quad \left[(\lambda+2\mu)\frac{d}{dr}\left(\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r}\right)+i(i+1)\frac{\mu}{r^2}\left(\frac{d(rv)}{dr}-u\right)\right]S_i=-\rho F_r,$$

$$(133)\quad \left[(\lambda+2\mu)\frac{1}{r}\left(\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r}\right)+\frac{\mu}{r}\frac{d}{dr}\left(\frac{d(rv)}{dr}-u\right)\right]\frac{\partial S_i}{\partial\theta}=-\rho F_\theta,$$

$$(134)\quad \left[(\lambda+2\mu)\frac{1}{r}\left(\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r}\right)+\frac{\mu}{r}\frac{d}{dr}\left(\frac{d(rv)}{dr}-u\right)\right]\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi}=-\rho F_\phi.$$

(c) *Bodily forces.* We now proceed to express the values of (1) the disturbing forces, (2) the gravitational force due to the change of density of the attracted element, (3) the attraction due to the change of configuration of the attracting mass.

(1) Disturbing forces. Let the potential of the disturbing forces be

$$(135)\quad W_i=-\frac{2}{3i}cga\frac{r^i}{a^i}S_i;$$

then the terms in ρF_r, ρF_θ, ρF_ϕ representing these forces are

$$(136)\quad \rho\frac{\partial W_i}{\partial r}=-\tfrac{2}{3}c\rho ga\frac{r^{i-1}}{a^i}S_i,$$

$$(137)\quad \frac{\rho}{r}\frac{\partial W_i}{\partial\theta}=-\frac{2}{3i}c\rho ga\frac{r^{i-1}}{a^i}\frac{\partial S_i}{\partial\theta},$$

$$(138)\quad \frac{\rho}{r\sin\theta}\frac{\partial W_i}{\partial\phi}=-\frac{2}{3i}c\rho ga\frac{r^{i-1}}{a^i}\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi}.$$

(2) Attraction of original mass-configuration upon increment of mass of element. The attraction of the original mass-configuration per unit mass upon an element distant r from the center is

$$-g\frac{r}{a}.$$

The increment of density is

$$(139)\quad -\rho\Delta=-\rho\left[\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r}\right]S_i.$$

Hence the attraction per unit volume, due to the increment of density of the attracted element, is

$$(140)\quad \begin{aligned}\left(-g\frac{r}{a}\right)(-\rho\Delta)&=g\rho\frac{r}{a}\left[\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r}\right]S_i\\&=g\rho a\frac{r}{a^2}\left[\frac{du}{dr}+2\frac{u}{r}-i(i+1)\frac{v}{r}\right]S_i.\end{aligned}$$

(3) Attraction due to increment of density of attracting mass. The potential at a point (r, θ, ϕ) due to a shell of density $D'S_i$, radius r' and thickness dr' is

$$\frac{4\pi\gamma}{2i+1}\frac{r^{i+2}}{r'^{i+1}}D'S_i dr' \qquad (r>r'), \tag{141}$$

$$\frac{4\pi\gamma}{2i+1}\frac{r^{i}}{r'^{i-1}}D'S_i dr' \qquad (r<r'). \tag{142}$$

On replacing $D'S_i$ by

$$-\rho\Delta' = \rho\left[i(i+1)\frac{v'}{r'^2} - \frac{1}{r'^2}\frac{d(r'^2u')}{dr'}\right]S_i,$$

the potential due to the change of density of the sphere is given by the expression

$$\frac{3gS_i}{(2i+1)a}\left[\frac{1}{r^{i+1}}\int_0^r\left(i(i+1)\frac{v'}{r'^2} - \frac{1}{r'^2}\frac{d(r'^2u')}{dr'}\right)r'^{i+2}dr' + r^i\int_r^a\left(i(i+1)\frac{v'}{r'^2} - \frac{1}{r'^2}\frac{d(r'^2u')}{dr'}\right)\frac{dr'}{r'^{i-1}}\right]. \tag{143}$$

To this must be added the potential due to the surface inequality of density ρ and thickness $u_1 S_i$:

$$\frac{3g}{2i+1}u_1\frac{r^i}{a^i}S_i. \tag{144}$$

Adding (143) and (144), and reducing by integrating by parts the last term in each definite integral, we see that the total potential may be expressed in the form

$$V_i = \frac{3gS_i}{(2i+1)a}\left[\frac{i}{r^{i+1}}\int_0^r\left((i+1)\frac{v'}{r'^2} + \frac{u'}{r'^2}\right)r'^{i+2}dr' + (i+1)r^i\int_r^a\left(i\frac{v'}{r'^2} - \frac{u'}{r'^2}\right)\frac{dr'}{r'^{i-1}}\right]. \tag{145}$$

The components of bodily force due to this potential are (per unit volume)

$$\rho\frac{\partial V_i}{\partial r} = \frac{3g\rho}{(2i+1)a}\left[-\frac{i(i+1)}{r^{i+2}}\int_0^r\left((i+1)\frac{v'}{r'^2} + \frac{u'}{r'^2}\right)r'^{i+2}dr' + i(i+1)r^{i-1}\int_r^a\left(i\frac{v'}{r'^2} - \frac{u'}{r'^2}\right)\frac{dr'}{r'^{i-1}} + (2i+1)u\right]S_i, \tag{146}$$

$$\frac{\rho}{r}\frac{\partial V_i}{\partial\theta} = \frac{3g\rho}{(2i+1)a}\left[\frac{i}{r^{i+2}}\int_0^r\left((i+1)\frac{v'}{r'^2} + \frac{u'}{r'^2}\right)r'^{i+2}dr' + (i+1)r^{i-1}\int_r^a\left(i\frac{v'}{r'^2} - \frac{u'}{r'^2}\right)dr'\right]\frac{\partial S_i}{\partial\theta}, \tag{147}$$

$$\frac{\rho}{r\sin\theta}\frac{\partial V_i}{\partial\phi}=\frac{3g\rho}{(2i+1)a}\left[\frac{i}{r^{i+2}}\int_0^r\left((i+1)\frac{v'}{r'}+\frac{u'}{r'}\right)r'^{i+2}dr'\right.$$

(148)

$$\left.+(i+1)r^{i-1}\int_r^a\left(i\frac{v'}{r'}-\frac{u'}{r'}\right)dr'\right]\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi}.$$

(4) Components of total bodily force. Combining all the bodily forces, we find

$$\rho F_r=\rho ga\left\{-\tfrac{2}{3}c\frac{r^{i-1}}{a^i}+\frac{r}{a^2}\left(\frac{du}{dr}+2\frac{u}{r}-i(i+1)\frac{v}{r}\right)+3\frac{u}{a^2}\right.$$

$$+\frac{3i(i+1)}{(2i+1)a^2}\left[-\frac{1}{r^{i+2}}\int_0^r\left((i+1)\frac{v'}{r'}+\frac{u'}{r'}\right)r'^{i+2}dr'\right.$$

(149)

$$\left.\left.+r^{i-1}\int_r^a\left(i\frac{v'}{r'}-\frac{u'}{r'}\right)\frac{dr'}{r'^{i-1}}\right]\right\}S_i,$$

$$\rho F_\theta=\rho ga\left\{-\frac{2}{3i}c\frac{r^{i-1}}{a^i}+\frac{3}{(2i+1)a^2}\left[\frac{i}{r^{i+2}}\int_0^r\left((i+1)\frac{v'}{r'}+\frac{u'}{r'}\right)r'^{i+2}dr'\right.\right.$$

(150)

$$\left.\left.+(i+1)r^{i-1}\int_r^a\left(i\frac{v'}{r'}-\frac{u'}{r'}\right)dr'\right]\right\}\frac{\partial S_i}{\partial\theta},$$

while ρF_ϕ differs from ρF_θ only in the substitution of $1/\sin\theta\cdot\partial S_i/\partial\phi$ for $\partial S_i/\partial\theta$. These values substituted in (132), (133) and (134) give the three equations of equilibrium for a volume element. It is seen that the factors involving θ and ϕ cancel, also that the second and third equations are identical. Hence the problem is reduced to the solution of two simultaneous ordinary differential equations for determining u and v in terms of r.

(*d*) *Final form of equations of interior equilibrium.* We now get the equations in final form by substituting in (132) and (133) the above values of ρF_r and ρF_θ. In writing the equations we shall replace u and v by new functions

$$e=\frac{u}{r},\qquad \alpha=\frac{v}{r},\tag{151}$$

and introduce as independent variable $x=r/a$. The equations then take the following forms:

$$(\lambda+2\mu)x^2\frac{d^2e}{dx^2}+4(\lambda+2\mu)x\frac{de}{dx}-i(i+1)\mu e-i(i+1)(\lambda+\mu)x\frac{d\alpha}{dx}$$

$$+2i(i+1)\mu\alpha+\rho ga\left[-\tfrac{2}{3}cx^i+\left(x\frac{de}{dx}+6e-i(i+1)\alpha\right)x^2\right.\tag{152}$$

$$\left.+\frac{3i(i+1)}{2i+1}\left(-\frac{1}{x^{i+1}}\int_0^x[(i+1)\alpha'+e']x'^{i+2}dx'+x^i\int_x^1(i\alpha'-e')\frac{dx'}{x'^{i-1}}\right)\right]=0,$$

$$(\lambda+\mu)x\frac{de}{dx}+(3\lambda+5\mu)e+\mu x^2\frac{d^2\alpha}{dx^2}+4\mu x\frac{d\alpha}{dx}-[i(i+1)\lambda+2(i^2+i-1)\mu]\alpha$$

$$+\rho g a\left[-\frac{2c}{3i}x^i+\frac{3}{2i+1}\left(\frac{i}{x^{i+1}}\int_0^x[(i+1)\alpha'+e']x'^{i+2}dx'\right.\right. \tag{153}$$

$$\left.\left.+(i+1)x^i\int_x^1(i\alpha'-e')\frac{dx'}{x'^{i-1}}\right)\right]=0.$$

§ 3. *Surface conditions.*

The stress upon the surface of the actual strained body is assumed to be 0. This is equivalent to assuming the body to extend to or beyond the original boundary $r=a$ in every direction, and assuming the stress on this spherical surface to be a normal stress equal to the weight of the surface inequality of thickness $u_1 S_i$ or $e_1 a S_i$, e_1 being the surface value of e.

That this is correct to the first order of small quantities may be shown by the reasoning outlined above in the solution of the case $i=2$, which need not be here repeated. Assuming its correctness, the boundary conditions are the following:

$$\left.\begin{aligned}\widehat{rr}&=-\rho g a e_1 S_i\\ \widehat{r\theta}&=0\\ \widehat{r\phi}&=0\end{aligned}\right\}\text{ when } x=1. \tag{154}$$

In these we must put

$$\widehat{rr}=\lambda\Delta+2\mu\frac{\partial u_r}{\partial r},$$

$$\widehat{r\theta}=\mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right), \tag{155}$$

$$\widehat{r\phi}=\mu\left(\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}+\frac{1}{r\sin\phi}\frac{\partial u_r}{\partial\phi}\right).$$

By means of (125), (126), (127) and (151) these take the forms,

$$\widehat{rr}=\left[(\lambda+2\mu)x\frac{de}{dx}+(3\lambda+2\mu)e-i(i+1)\lambda\alpha\right]S_i,$$

$$\widehat{r\theta}=\mu\left(x\frac{d\alpha}{dx}+e\right)\frac{\partial S_i}{\partial\theta}, \tag{156}$$

$$\widehat{r\phi}=\mu\left(x\frac{d\alpha}{dx}+e\right)\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi},$$

so that the three equations (154) reduce to the two equations,

$$(157) \qquad (\lambda + 2\mu)x\frac{de}{dx} + (3\lambda + 2\mu)e - i(i+1)\lambda\alpha = -\rho g a e,$$

$$(158) \qquad x\frac{d\alpha}{dx} + e = 0,$$

which must be satisfied for $x = 1$.

[It is easy to see what equations must replace these if different surface conditions are given; for example, if there are known surface stresses proportional to S_i, $\partial S_i/\partial\theta$, $1/\sin\theta \cdot \partial S_i/\partial\phi$, or given surface displacements of a similar type.]

§ 4. *Solution of equations.*

A solution of (152) and (153) satisfying (157) and (158) is obtained by assuming

$$(159) \qquad e = \sum A_m x^m, \qquad \alpha = \sum B_m x^m.$$

The substitution of $A_m x^m$, $B_m x^m$ for e and α yields in (152) the terms

$$(160) \quad \begin{aligned} &\left\{[(\lambda+2\mu)(m^2+3m)-\mu i(i+1)]A_m - i(i+1)[(\lambda+\mu)m-2\mu]B_m\right\}x^m \\ &+\rho g a\left\{[(m+6)A_m - i(i+1)B_m]x^{m+2}\right. \\ &\left.+\frac{3i(i+1)}{2i+1}\left[-\frac{A_m+(i+1)B_m}{m+i+3}x^{m+2}+\frac{A_m-iB_m}{m-i+2}x^{m+2}-\frac{A_m-iB_m}{m-i+2}x^i\right]\right\}, \end{aligned}$$

and in (153) the terms

$$(161) \quad \begin{aligned} &\left\{[(\lambda+\mu)m+(3\lambda+5\mu)]A_m+[\mu(m^2+3m)-(i(i+1)\lambda+2(i^2+i-1)\mu)]B_m\right\}x^m \\ &+\rho g a\frac{3}{2i+1}\left[\frac{i[A_m+(i+1)B_m]}{m+i+3}x^{m+2}\right. \\ &\left.+\frac{(i+1)(A_m-iB_m)}{m-i+2}x^{m+2}-\frac{(i+1)(A_m-iB_m)}{m-i+2}x^i\right]. \end{aligned}$$

Notice first that the terms in x^i in these two expressions are in the same ratio as the original terms of degree i in (152) and (153); therefore, the terms of degree i in both equations can be made to vanish by a single relation among the coefficients in the assumed series (159).

Again, for the terms of lowest degree, the coefficients of x^m in (160) and (161) must vanish, and this is possible without the vanishing of A_m and B_m only if the determinant of the coefficients of A_m and B_m vanishes. The value of this determinant is found to be

$$(162) \qquad (\lambda + 2\mu)\mu(m-i)(m-i+2)(m+i+1)(m+i+3).$$

The four values of m for which this vanishes, with the corresponding values of B_m/A_m, are as follows:

m	$\frac{B_m}{A_m} = \frac{(\lambda+2\mu)(m^2+3m)-\mu i(i+1)}{i(i+1)[(\lambda+\mu)m-2\mu]}$
i	$\frac{(i+3)\lambda+(i+5)\mu}{(i+1)[i\lambda+(i-2)\mu]}$
$i-2$	$\frac{1}{i}$
$-i-1$	$-\frac{(i-2)\lambda+(i-4)\mu}{i(i+1)\lambda+i(i+3)\mu}$
$-i-3$	$-\frac{1}{i+1}$

Negative powers of x being excluded by physical considerations, the series may start with terms of degree $i-2$, provided i is not less than 2. If i is positive and less than 2, the lowest degree available would be i.

We here assume i equal to or greater than 2, and let the series (159) take the forms

$$\begin{aligned} e &= A_{i-2}x^{i-2} + A_i x^i + A_{i+2}x^{i+2} + \cdots, \\ \alpha &= \frac{1}{i}A_{i-2}x^{i-2} + B_i x^i + B_{i+2}x^{i+2} + \cdots. \end{aligned} \tag{163}$$

From (160) and (161) it is seen that (excluding the terms of degree i) the coefficients of x^{m+2} resulting from the substitution of (163) in (152) and (153) will vanish if the following equations are satisfied:

$$\begin{aligned} &[(\lambda+2\mu)(m+2)(m+5)-\mu i(i+1)]A_{m+2} - i(i+1)[(\lambda+\mu)(m+2)-2\mu]B_{m+2} \\ &\quad + \rho g a \frac{m^2+8m-(i-3)(i+4)}{(m+i+3)(m-i+2)}[(m+3)A_m - i(i+1)B_m] = 0, \end{aligned} \tag{164}$$

$$\begin{aligned} &[(\lambda+\mu)(m+2)+3\lambda+5\mu]A_{m+2} \\ &\quad + \left\{\mu(m+2)(m+5) - [i(i+1)\lambda + 2(i^2+i-1)\mu]\right\}B_{m+2} \\ &\quad + \rho g a \frac{3}{(m+i+3)(m-i+2)}[(m+3)A_m - i(i+1)B_m] = 0. \end{aligned} \tag{165}$$

By means of these equations the coefficients $A_{i+2}, A_{i+4}, \cdots, B_{i+2}, B_{i+4}, \cdots$ may be expressed in terms of A_i, B_i, leaving thus far arbitrary the three constants A_{i-2}, A_i, B_i.

It remains only to equate to zero the coefficients of x^i, which as already

remarked requires but a single equation. The two differential equations will thus be satisfied by values of e and α of the assumed form, containing two arbitrary constants, provided the series thus determined prove to be convergent. The two arbitrary constants suffice for the satisfaction of the surface conditions.

The solution of (164) and (165) gives

$$A_{m+2} = -\frac{\rho g a}{\lambda+2\mu} \times \frac{[(m+4)(m+6)\mu - i(i+1)(\lambda+2\mu)][(m+3)A_m - i(i+1)B_m]}{(m-i+2)(m+i+5)(m-i+4)(m+i+3)\mu}, \tag{166}$$

$$B_{m+2} = \frac{\rho g a[m(\lambda+\mu)+5\lambda+4\mu][(m+3)A_m - i(i+1)B_m]}{(\lambda+2\mu)(m-i+2)(m+i+5)(m-i+4)(m+i+3)\mu}, \tag{167}$$

$$\frac{(m+5)A_{m+2} - i(i+1)B_{m+2}}{(m+3)A_m - i(i+1)B_m} = -\frac{m+6}{(m-i+2)(m+i+3)} \cdot \frac{\rho g a}{\lambda+2\mu}. \tag{168}$$

By means of these equations the values of all coefficients after the first two in each of the assumed series (163) may be expressed in terms of A_i and B_i. These values may be expressed as follows, n taking values 0, 1, 2, $\cdots$:

$$A_{i+2n+2} = (-b)^{n+1}\left[i(i+1)\frac{\lambda}{\mu} + 2i(i+1) - (i+2n+4)(i+2n+6)\right] \times \frac{(i+4)(i+6)\cdots(i+2n+4)[i(i+1)B_i - (i+3)A_i]}{(i+4)[2\cdot 4\cdots(2n+4)][(2i+3)(2i+5)\cdots(2i+2n+5)]}, \tag{169}$$

$$B_{i+2n+2} = (-b)^{n+1}\left[(i+2n+5)\frac{\lambda}{\mu} + i + 2n + 4\right] \times \frac{(i+4)(i+6)\cdots(i+2n+4)[i(i+1)B_i - (i+3)A_i]}{(i+4)[2\cdot 4\cdots(2n+4)][(2i+3)(2i+5)\cdots(2i+2n+5)]}. \tag{170}$$

The coefficient of x^i may be found by substituting in (160) or (161) successively $m = i-2, i, i+2, \cdots$, remembering that $iB_{i-2} = A_{i-2}$, and that, as already pointed out, a single equation causes the vanishing of the coefficients of x^i in both (152) and (153). This equation may be written in the following form, after dividing through by $\lambda + 2\mu$ and writing b for $g\rho a/(\lambda+2\mu)$:

$$i(i+1)\left[\left(\frac{\lambda+\mu}{\lambda+2\mu} + \frac{2}{i+1} - \frac{3b}{2(2i+1)}\right)A_i - \left(\frac{i\lambda+(i-2)\mu}{\lambda+2\mu} - \frac{3ib}{2(2i+1)}\right)B_i\right] + \frac{3ib}{2i+1}A_{i-2} - \frac{3i(i+1)b}{2i+1}\sum_0^\infty \frac{A_{i+2+2n} - iB_{i+2+2n}}{2n+4} - \tfrac{2}{3}bc = 0. \tag{171}$$

Also, from (169) and (170),

$$\frac{A_{i+2+2n}-iB_{i+2+2n}}{2n+4}=(-b)^{n+1}\left[i\frac{\lambda}{\mu}+3i+2(n+3)\right]$$
$$\times\frac{[(i+4)(i+6)\cdots(i+2n+4)]\,[(i+3)A_i-i(i+1)B_i]}{(i+4)[2\cdot4\cdots(2n+4)]\,[(2i+3)(2i+5)\cdots(2i+2n+5)]}, \tag{172}$$

by means of which (171) is reduced to a relation among the three constants A_{i-2}, A_i, B_i.

The differential equations (152) and (153) are thus satisfied by (163), provided the coefficients satisfy (169), (170), (171) and (172); and two of the coefficients still remain arbitrary. By properly assigning these, the surface conditions (157) and (158) may be satisfied. The solution may be expressed in fairly compact form by means of the following functions:

$$L(z)=\sum_0^\infty(-1)^n\left[i(i+1)\frac{\lambda}{\mu}+2i(i+1)-(i+2n+4)(i+2n+6)\right]$$
$$\times\frac{(i+4)(i+6)\cdots(i+2n+4)}{(i+4)[2\cdot4\cdots(2n+4)]\,[(2i+3)(2i+5)\cdots(2i+2n+5)]}z^n,$$

$$M(z)=\sum_0^\infty(-1)^n\left[(i+2n+5)\frac{\lambda}{\mu}+i+2n+4\right]$$
$$\times\frac{(i+4)(i+6)\cdots(i+2n+4)}{(i+4)[2\cdot4\cdots(2n+4)]\,[(2i+3)(2i+5)\cdots(2i+2n+5)]}z^n,$$

$$N(z)=\sum_0^\infty(-1)^n\left[i\frac{\lambda}{\mu}+3i+2(n+3)\right]$$
$$\times\frac{(i+4)(i+6)\cdots(i+2n+4)}{(i+4)[2\cdot4\cdots(2n+4)]\,[(2i+3)(2i+5)\cdots(2i+2n+5)]}z^n, \tag{173}$$

$$L'(z)=\sum_0^\infty(-1)^n\left[i(i+1)\frac{\lambda}{\mu}+2i(i+1)-(i+2n+4)(i+2n+6)\right]$$
$$\times\frac{(i+2n+2)[(i+4)(i+6)\cdots(i+2n+4)]}{(i+4)[2\cdot4\cdots(2n+4)]\,[(2i+3)(2i+5)\cdots(2i+2n+5)]}z^n,$$

$$M'(z)=\sum_0^\infty(-1)^n\left[(i+2n+5)\frac{\lambda}{\mu}+i+2n+4\right]$$
$$\times\frac{(i+2n+2)[(i+4)(i+6)\cdots(i+2n+4)]}{(i+4)[2\cdot4\cdots(2n+4)]\,[(2i+3)(2i+5)\cdots(2i+2n+5)]}z^n.$$

The values of e and α, and the three equations for determining A_{i-2}, A_i and B_i, may then be written as follows, the third equation being equivalent to (171), the fourth to (157), and the fifth to (158):

$$(A)\quad e = A_{i-2}x^{i-2} + A_i x^i + [(i+3)A_i - i(i+1)B_i]\,bx^{i+2}L(bx^2),$$

$$(B)\quad \alpha = \frac{1}{i}A_{i-2}x^{i-2} + B_i x^i + [(i+3)A_i - i(i+1)B_i]bx^{i+2}M(bx^2),$$

$$(C)\quad \begin{aligned}&\frac{3ib}{2i+1}(A_{i-2}+A_i) + \frac{(4i+6)\mu}{\lambda+2\mu}A_i\\ &+[(i+3)A_i - i(i+1)B_i]\left(\frac{i\lambda+(i-2)\mu}{\lambda+2\mu} - \frac{3ib}{2(2i+1)} + \frac{3i(i+1)}{2i+1}b^2N(b)\right) - \tfrac{2}{3}bc = 0,\end{aligned}$$

$$(D)\quad \begin{aligned}&\left[b\left(\frac{\lambda}{\mu}+2\right)+2i-2\right](A_{i-2}+A_i) + 4A_i + [(i+3)A_i - i(i+1)B_i]\\ &\times\left[\frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}+2\right)bL'(b) + \left(3\frac{\lambda}{\mu}+2\right)bL(b)\right.\\ &\left.+\left(\frac{\lambda}{\mu}+2\right)b^2L(b) - i(i+1)\frac{\lambda}{\mu}bM(b)\right] = 0,\end{aligned}$$

$$(E)\quad \begin{aligned}&(i^2-1)(A_{i-2}+A_i) + (2i+1)A_i + \frac{i}{2}[(i+3)A_i - i(i+1)B_i]\\ &\times[-1 + (i+1)bL(b) + (i+1)bM'(b)] = 0.\end{aligned}$$

These correspond to equations (A), (B), (C), (D), (E) obtained in Part I of this paper, and reduce to them when $i = 2$. It must be noted, however, that the notation is slightly different, e, α and the functions (173) differing by constant factors from the corresponding functions in Part I.